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A THEORETICAL INVESTIGATION OF THE SCHRÖDINGER EQUATION FOR A MODIFIED TWO DIMENSIONAL PURELY SEXTIC DOUBLE-WELL POTENTIAL

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ABSTRACT

A theoretical analytical investigation for the exact solvability of non-relativistic quantum spectrum systems, at low energy for modified two-dimensional purely sextic double-well potential (2D-modified sextic DWAO potential) is discussed, by means Bopp's shift method, instead to solving deformed Schrödinger equation with star product, in the framework of both noncommutativite two dimensional real space and phase (NC: 2D-RSP), the exact corrections for lowest excitations are found straightforwardly for interactions in one-electron atoms by means of the standard perturbation theory for three special cases. Furthermore, the obtained corrections of energies are depended on the four infinitesimals parameters (θ , χ) and ($\overline{\theta}$, $\overline{\sigma}$), which are induced by position-position and momentum-momentum noncommutativity, respectively, in addition to the discreet atomic quantum numbers ($j = l \pm 1/2, s = \pm 1/2, l$) and the magnetic quantum number m . We have also generalized our obtained results to include the other atoms with spin $\vec{S} \neq 1/\vec{2}$.

Keywords: *Two-dimensional purely sextic double-well potential, Noncommutative space and phase, Star product and Bopp's shift method*

1. INTRODUCTION

The double-well anharmonic oscillator (DWAO) holds a great importance because of its relation to many problems of quantum chemistry and field theory [1-4]. Dai-Nam Le et *al*. was studied the Schrödinger equation for the purely sextic double-well potential problem in two-dimensional space by wave function ansatz method for analytical approach and by Feranchuk-Komarov operator method for numerical approach [1]. Over the past few years, many efforts have been produced to study quantum systems on based to the Dirac equation, Klein-Gordon equation and Schrödinger equation, which is undoubtedly the most widely studied equation of modern physics, for spherical and non spherical potentials that in ordinary space of quantum mechanics and its extension, which known by extended quantum mechanics. The noncommutativity was introduced firstly by Heisenberg W. in 1930 [5] and then by Syndre H. in 1947 [6]. Recently, physicists have developed previously works by applying the noncommutativity properties on space and phase, the two ordinary generalized coordinates of space and momentums $\left(x^{\mu}, p^{\mu}\right)$ replaced by corresponding new values $\left(\hat{x}^{\mu}, \hat{p}^{\mu}\right)$, thus the two ordinary commutation $\left[x^{\mu}, x^{\nu}\right]$

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and $\left[p^{\mu}, p^{\nu}\right]$ will be replace by two new commutation relations in NCQM $\left[\hat{x}^{\mu}, \hat{x}^{\nu}\right]_{*}$ and $\left[\hat{p}^{\mu}, \hat{p}^{\nu}\right]_{*}$ respectively (in natural units) [7-25]:

$$\left[\hat{x}^{\mu}, \hat{x}^{\nu}\right]_{*} = i\theta^{\mu\nu} \text{ and } \left[\hat{p}^{\mu}, \hat{p}^{\nu}\right]_{*} = i\bar{\theta}^{\mu\nu} \quad (1)$$

The symbol $(*)$ and the two parameters $\left(\theta^{\mu\nu}, \bar{\theta}^{\mu\nu}\right) \equiv \varepsilon^{\mu\nu}\left(\theta, \bar{\theta}\right)$ are star product and antisymmetric real tensors induced by position-position and momentum-momentum noncommutativity, respectively, it's important to notice that, the above two fundamental commutation relations are satisfied in particulars' cases from the general star product $f(x, p) * g(x, p)$ between two arbitrary functions $f(x, p)$ and $g(x, p)$ in the first order of two parameters $\theta^{\mu\nu}$ and $\bar{\theta}^{\mu\nu}$ as follow [18-32]:

$$(f * g)(x, p) = (fg)(x, p) - \frac{i}{2}\theta^{\mu\nu}\frac{\partial f(x, p)}{\partial x^{\mu}}\frac{\partial g(x, p)}{\partial x^{\nu}} - \frac{i}{2}\bar{\theta}^{\mu\nu}\frac{\partial f(x, p)}{\partial p^{\mu}}\frac{\partial g(x, p)}{\partial p^{\nu}} \quad (2)$$

The first term is the usual product in commutative space while the rest two parts are represents the effects of the noncommutativity of (space-space) and (phase-phase), respectively; a Bopp's shift method can be used, instead of solving any quantum systems by using directly star product procedure [28-34]:

$$\left[\hat{x}_i, \hat{x}_j\right] = i\theta_{ij} \text{ and } \left[\hat{p}_i, \hat{p}_j\right] = i\bar{\theta}_{ij} \quad (3)$$

The generalized positions and momentum coordinates in the new 2D-NCQM $(\hat{x}, \hat{y})$ and $(\hat{p}_x, \hat{p}_y)$ generalized coordinates and momentums are depended with corresponding usual generalized positions and momentums coordinates (x, y) and (p_x, p_y), in the 2D-QM, by the following relations [27-36]:

$$\begin{aligned} &\hat{x}_i = x_i - \frac{\theta_{ij}}{2}p_j \Rightarrow \hat{x} = x - \frac{\theta}{2}p_y \text{ and } \quad \hat{y} = y + \frac{\theta}{2}p_x \\ &\hat{p}_i = p_i + \frac{\bar{\theta}_{ij}}{2}x_j \Rightarrow \hat{p}_x = p_x + \frac{\bar{\theta}}{2}y \text{ and } \quad \hat{p}_y = p_x - \frac{\bar{\theta}}{2}x \end{aligned} \quad (4)$$

Which allow us to getting the two operators $\hat{r}^2$ and $\hat{p}^2$ on a (NC: 2D-RSP) as follows [28-36, 38-42]:

$$\left(r^2, p^2\right) \Rightarrow \left(\hat{r}^2, \hat{p}^2\right) = \left(r^2 - \theta L_z, p^2 + \bar{\theta}L_z\right) \quad (5)$$

With $L_z \equiv xp_y - yp_x$, this work is aimed at obtaining an analytic expression for the eigenenergies of a 2D-modified sextic DWAO potential in (NC: 2D-RSP) using the generalization Bopp's shift method based on mentioned formalisms on above equations to discover the new symmetries and a possibility to obtain another applications to this potential in different fields. The contents of the rest paper are as follows: In next section, we briefly review the Schrödinger equation with 2D-sextic DWAO potential. The Section 3, reserved to derive the 2D-modified sextic DWAO potential and perturbative terms of Hamiltonian operator using generalized Bopp's shift method for modified 2D-sextic DWAO potential. In the fourth section, we apply independent time standard perturbation theory to find the quantum spectrum of the lowest excitations in (NC: 2D RSP) for spin-orbital interaction corresponding three special cases. In the next section, we deduce the new magnetic spectrum operator for studied potential. In the sixth section, we resume the global spectrum and corresponding noncommutative Hamiltonian for 2D-modified sextic DWAO potential. In the seventh section, we extended our study to include the energy spectra for other atoms with spin $\vec{S} \neq 1/\vec{2}$ in (NC: 2D RSP). Finally, the important results and the conclusions are given in last section.

2. REVIEW THE EIGNENFUNCTIONS AND THE ENERGY EIGENVALUES FOR 2D-SEXTIC DWAO POTENTIAL

Let us present a brief review of time independent Schrödinger equation for a fermionic particle like electron of rest

mass M and its energy E moving in 2D-sextic DWAO potential [1]:

$$V(r) = -\frac{1}{2}M\omega^2\rho^2 + \frac{\lambda}{8}\rho^6 \qquad (6)$$

Here λ is an anharmonicity constant and the parameter ω is real and positive. In the polar coordinates (ρ,φ), the variables in the SE are separated and the wave functions can be presented in the form $\Psi(\rho,\varphi) = \frac{1}{\sqrt{2\prod}}R(\rho)\exp(im\varphi)$, with the magnetic quantum number ($m = 0,\pm1,\pm2,.....$), the radial wave function $R(\rho)$ is a solution of the following equation [1]:

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2}{\rho^2} + 2E + \beta\rho^2 - \frac{1}{4}\rho^6\right)R(\rho) = 0 \qquad (7)$$

where $\beta = \frac{M^{3/2}\omega^2}{\hbar\lambda^{1/2}}$ is dimensionless and is referred to as a geometry parameter of the well. The wave function $\Psi_{E,m}(\rho,\varphi)$ in a general form [1]:

$$\Psi_{E,m}(\rho,\varphi) = \frac{C_0}{\sqrt{2\prod}}\exp(im\varphi)\rho^{\mu}\exp\left(-\frac{\rho^4}{8}\right)f(\rho) \quad (8)$$

The function $f(\rho)$ is given by:

$$f(\rho) = \sum_{k=0}^{\infty}C_k\rho^{2k} \equiv \sum_{k=0}^{\infty}\frac{|m|!}{2^k k!(k+|m|)!}D_k\rho^{2k} \quad (9)$$

The coefficient C_0 is the normalization condition while $D_k = C_{2k}\frac{2^k k!(k+|m|)!}{C_0|m|!}$ with the coefficients C_{2k}. The function $f(\rho)$ is able to become a polynomial of the value ρ, when the geometry values β take a specific combined quantum number ($2N+2+|m|$), we are interested for study the following cases:

- The case of $N=0$ corresponds to the allowed geometry values $\beta = 2+|m|$ (describing the quantum state with principal quantum number $n = 1+|m|$) and the energy is equal to zero $E=0$, the wave function for this case is [1]:

$$\Psi_{n=1+|m|,m}(\rho,\varphi) = \frac{1}{\sqrt{2\prod}}\rho^{|m|}\exp\left(-\frac{\rho^4}{8}\right)\exp(im\varphi) \quad (10)$$

- For the case of $N=1$, the allowed geometry values are $\beta = 4+|m|$, there are two solutions for energy $E_{n,m}$ as follows [1]:

$$\Psi_{n=1+|m|,m}(\rho,\varphi) = \frac{1}{\sqrt{2\prod}}\rho^{|m|}\exp\left(-\frac{\rho^4}{8}\right)\left(1+\frac{\rho^2}{\sqrt{4|m|+2}}\right)\exp(im\varphi) \Rightarrow E_{n,m} = -\sqrt{2|m|+2}$$
$$\Psi_{n=2+|m|,m}(\rho,\varphi) = \frac{1}{\sqrt{2\prod}}\rho^{|m|}\exp\left(-\frac{\rho^4}{8}\right)\left(1-\frac{\rho^2}{\sqrt{4|m|+2}}\right)\exp(im\varphi) \Rightarrow E_{n,m} = \sqrt{2|m|+2} \qquad (11)$$

When $N\rangle1$, the geometry values β allowed more two values of principal quantum number, more two values of energy and more two values of waves functions (For more details please return to the ref. [1])

3. THEORY OF (NC-2D: RSP) FOR 2D-MODIFIED SEXTIC DWAO POTENTIAL

In this section, we shall study the 2D-modified sextic DWAO potential in (NC: 2D-RSP), to perform this task the physical form of NR-SE, we apply the following steps [22-32]:

1. Ordinary two dimensional Hamiltonian operators $\hat{H}_{dwao}(p_i, x_i)$ will be replace by new 2D Hamiltonian operators $\hat{H}_{nc-dwao}(\hat{p}_i, \hat{x}_i)$,

2. Ordinary complex wave function $\Psi(\vec{r})$ will be replacing by new complex wave function $\widehat{\Psi}(\overleftrightarrow{r})$,
3. Ordinary energy $E_{n,m}$ will be replaced by new values $E_{nc-dwao}$.

And the last step corresponds to replace the ordinary old product by new star product $(*)$, which allow us to constructing the 2D-DSE in both (NC-2D: RSP) for 2D-modified sextic DWAO potential:

$$\hat{H}_{nc-dwao}(\hat{p}_i,\hat{x}_i)*\widehat{\Psi}(\overleftrightarrow{r})=E_{nc-dwao}\widehat{\Psi}(\overleftrightarrow{r}) \quad (12)$$

It's important to notice that, the two obtained results in eq. (5) conserved the symmetry between ($\hat{p}_i$ and $\hat{x}_i$) and the rolls of θ and $\bar{\theta}$ are inversed $(\theta\approx-\bar{\theta})$ [36]. The only none null two commutations relations: $\{[\hat{x},\hat{y}]$ and $[\hat{p}_x,\hat{p}_y]\}$ and the two uncertainties relations $\{\Delta x\Delta y$ and $\Delta p_x\Delta p_y\}$ are determined from the following projections relations:

$$\begin{aligned}([\hat{x},\hat{y}],\quad [\hat{p}_x,\hat{p}_y])&=i(\theta,-\bar{\theta})\\ (\Delta x\Delta y\ ,\ \Delta p_x\Delta p_y)&\approx(\theta,\ \bar{\theta})\end{aligned} \quad (13)$$

here $(\theta,\bar{\theta})\equiv(\theta^{12},\bar{\theta}^{12})$, as mentioned before, we apply the Bopp's shift method on the equation (12) to obtain the reduced SE for 2D-modified sextic DWAO potential (Without star product):

$$H(\hat{p}_i,\hat{x}_i)\psi(\vec{r})=E_{nc-dwao}\psi(\vec{r}) \quad (14)$$

Where the new Hamiltonian operator $H(\hat{p}_i,\hat{x}_i)$ can be expressed in three general varieties: (NC-2D: RSP), (NC: 2D-RS) and (NC-2D: RP). In recently work, we are interest with the first variety (NC-2D: RSP):

$$H_{nc-dwao}(\hat{p}_i,\hat{x}_i)\equiv H\left(\hat{p}_x=p_x+\frac{\bar{\theta}}{2}y,\hat{p}_y=p_y-\frac{\bar{\theta}}{2}x,\hat{x}=x-\frac{\theta}{2}p_y,\hat{y}=y+\frac{\theta}{2}p_x\right)\text{for NC-2D:RSP} \quad (15)$$

After straightforward calculations, we can obtain the three important terms, which will be use to determine the 2D-modified sextic DWAO potential in (NC: 2D- RSP):

$$\begin{aligned}-\frac{1}{2}M\omega^2\rho^2&\to-\frac{1}{2}M\omega^2\hat{\rho}^2=-\frac{1}{2}M\omega^2\rho^2+\frac{1}{2}M\omega^2\theta L_z\\ \frac{\lambda}{8}\rho^6&\to\frac{\lambda}{8}\hat{\rho}^6=\frac{\lambda}{8}\rho^6-\frac{\lambda}{4}\rho^4\theta L_z\\ \frac{p^2}{2M}&\to\frac{\hat{p}^2}{2M}=\frac{p^2}{2M}+\frac{\bar{\theta}}{2M}L_z\end{aligned} \quad (16)$$

From above relations, one can write the formed the operator $V_{dwao}(\hat{\rho})$ for 2D-modified sextic DWAO potential and the noncommutative kinetic term $\frac{\hat{p}^2}{2M}$, respectively:

$$\begin{aligned}V_{dwao}(\hat{\rho})&=-\frac{1}{2}M\omega^2\hat{\rho}^2+\frac{\lambda}{8}\hat{\rho}^6\\ \frac{\hat{p}^2}{2M}&=-\frac{1}{2M}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right)+\frac{\bar{\theta}L_z}{2M}\end{aligned} \quad (17)$$

Which allow us to obtaining the global potential operator $H_{\text{nc-dwao}}(\hat{\rho})$ for 2D-modified sextic DWAO potential in (NC: 2D-RSP) as:

$$H_{nc-dwao}(\hat{\rho})=-\frac{1}{2M}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right)-\frac{1}{2}M\omega^2\rho^2+\frac{\lambda}{8}\rho^6+\left(\theta\left(\frac{1}{2}M\omega^2-\frac{\lambda}{4}\rho^4\right)+\frac{\bar{\theta}}{2M}\right)L_z \quad (18)$$

It's clearly, that the two first terms are given the ordinary 2D-sextic DWAO potential, while the rest terms are proportional's with <u>two infinitesimals parameters</u> (θ and $\bar{\theta}$) and then gives the terms of perturbations Hamiltonian operator $H_{\text{per-}dwao}(\rho)$ in (NC: 2D-RSP) as:

$$H_{\text{per-dwao}}(\rho)=\left(\theta\left(\frac{1}{2}M\omega^2-\frac{\lambda}{4}\rho^4\right)+\frac{\bar{\theta}}{2M}\right)L_z \quad (19)$$

It should be noted that there are many methods used in the solution of the Schrödinger equation, which related to the

non-perturbative description of quantum systems, but our case is in this research related to the framework of the perturbative quantum systems, for further reading you can go back to the book [37].

4. THE EXACT SPIN-ORBITAL SPECTRUM FOR 2D-MODIFIED SEXTIC DWAO POTENTIAL IN NC:2D-RSP) FOR ONE-ELECTRON ATOMS.

Again, the perturbative term $H_{\text{per-}dwao}(r)$ can be rewritten to the equivalent physical form for 2D-modified sextic DWAO potential, we have make the same physics procedure in our previous works [38-45]:

$$H_{\text{per-}dwao}(r)=\gamma\left(\theta\left(\frac{1}{2}M\omega^2-\frac{\lambda}{4}\rho^4\right)+\frac{\bar{\theta}}{2M}\right)\vec{L}\vec{S} \quad (20)$$

Here γ is the fine structure constante. The above result is logical for two reasons: the first one $\vec{S}=1/2$ (in one-electron atoms) and the second $\left(L_x=L_y=0\right)$, furthermore, the above perturbative term $H_{\text{per-}dwao}(\rho)$ can be rewritten to the following new equivalent form for 2D-modified sextic DWAO potential:

$$H_{\text{per-}dwao}(\rho)=\gamma\left(\theta\left(\frac{1}{2}M\omega^2-\frac{\lambda}{4}\rho^4\right)+\frac{\bar{\theta}}{2M}\right)G^2 \quad (21)$$

with $G^2\equiv\vec{J}^2-\vec{L}^2-\vec{S}^2$. To the best of our knowledge, we just replace the coupling spin-orbital $\vec{L}\vec{S}$ by the expression $\frac{1}{2}\left(\vec{J}^2-\vec{L}^2-\vec{S}^2\right)$, in quantum mechanics. The set ($H_{\text{so-}dwao}(\rho)$, J^2, L^2, S^2 and J_z) forms a complete of conserved physics quantities and the eigenvalues of the spin orbital coupling operator are $k_{\pm}\left(j=l\pm1/2,l,s=1/2\right)\equiv\frac{1}{2}\left\{\left(l\pm\frac{1}{2}\right)\left(l\pm\frac{1}{2}+1\right)+l(l+1)-\frac{3}{4}\right\}$ corresponding $j=l+\frac{1}{2}$ (spin up) and $j=l-\frac{1}{2}$ (spin down), respectively, which allows us to form a diagonal (2×2) matrix, with non-null elements are $\left(H^{so}_{dwao}\right)_{11}$ and $\left(H^{so}_{dwao}\right)_{22}$ for 2D-modified sextic DWAO potential in (NC: 2D-RSP) as:

$$\left(H^{so}_{dwao}\right)_{11}=k_{+}\gamma\left(\theta\left(\frac{1}{2}M\omega^2-\frac{\lambda}{4}\rho^4\right)+\frac{\bar{\theta}}{2M}\right)\text{if } j=l+\tfrac{1}{2}\Rightarrow\text{spin-up}$$

$$\left(H^{so}_{dwao}\right)_{22}=k_{-}\gamma\left(\theta\left(\frac{1}{2}M\omega^2-\frac{\lambda}{4}\rho^4\right)+\frac{\bar{\theta}}{2M}\right)\text{ if } j=l-\tfrac{1}{2}\Rightarrow\text{spin-down} \quad (22)$$

the variables in the DSE, which presented in eq. (14), are separated and the wave functions can be presented in the form $\Psi(\rho,\varphi)=\frac{1}{\sqrt{2\Pi}}R(\rho)\exp(im\varphi)$, the new radial wave function $R(\rho)$ is a solution of the following equation:

$$\left(\frac{d^2}{d\rho^2}+E_{nc-dwao}+\frac{1}{2}M\omega^2\rho^2-\frac{\lambda}{8}\rho^6-\gamma\left(\theta\left(\frac{1}{2}M\omega^2-\frac{\lambda}{4}\rho^4\right)+\frac{\bar{\theta}}{2M}\right)\vec{L}\vec{S}-\frac{m^2-\frac{1}{4}}{\rho^2}\right)R(r)=0 \quad (23)$$

It is clearly that the above equation including equation (20), the perturbative terms of Hamiltonian operator, which we are subject of discussion in next subsection, we look for a possibility to getting an exact solution of equation (23) including the added new part $H_{\text{per-}dwao}(r)$.

4.1 THE EXACT SPIN-ORBITAL SPECTRUM FOR 2D-MODIFIED SEXTIC DWAO POTENTIAL IN (NC: 2D- RSP) FOR ONE-ELECTRON ATOMS

In this subsection, we are going to study the modifications to the energy levels ($E^{(N)}_{\text{u-}dwao}$ and $E^{(N)}_{\text{u-}dwao}$) for spin up and spin down, respectively, at first order of two parameters (θ and $\bar{\theta}$) obtained by applying the standard perturbation theory, because the added new part $H_{\text{per-}dwao}(r)$, is very small compared to the principal part of 2D-modified sextic DWAO potential, which presented in eq. (6). We are interested to treat the following three principle cases:

4.1.1 STUDY THE CASE $N=0$ AND GEOMETRY VALUES $\beta=2+|m|$ WITH PRINCIPAL QUANTUM NUMBER $n=1+|m|$

By applying the two axioms related to the energy

$$E^{(1)}_{\text{u-}dwao}=\left\langle \Psi_{n=1+|m|,m}(\rho,\varphi)\left(H^{so}_{dwao}\right)_{11}\Psi_{n=1+|m|,m}(\rho,\varphi)\right\rangle$$

$$E^{(1)}_{\text{d-}dwao}=\left\langle \Psi_{n=1+|m|,m}(\rho,\varphi)\left(H^{so}_{dwao}\right)_{22}\Psi_{n=1+|m|,m}(\rho,\varphi)\right\rangle$$

and using the wave equation in eq. (11), we may obtain the modifications to the energy levels for 2D-modified sextic DWAO potential $\left(E^{(0)}_{\text{u-}dwao},E^{(0)}_{\text{d-}dwao}\right)$ for the case $N=1$ and geometry values $\beta=4+|m|$ with principal quantum number $n=1+|m|$:

$$E^{(0)}_{\text{u-}dwao}=|N_0|^2\gamma_+\left(j=l+1/2,l,s=1/2\right)\int_0^{+\infty}\rho^{2|m|+1}\exp\left(-\frac{\rho^4}{4}\right)\left(\theta\left(\frac{1}{2}M\omega^2-\frac{\lambda}{4}\rho^4\right)+\frac{\bar{\theta}}{2M}\right)d\rho$$

$$E^{(0)}_{\text{d-}dwao}=|N_0|^2\gamma_-\left(j=l+1/2,l,s=1/2\right)\int_0^{+\infty}\rho^{2|m|+1}\exp\left(-\frac{\rho^4}{4}\right)\left(\theta\left(\frac{1}{2}M\omega^2-\frac{\lambda}{4}\rho^4\right)+\frac{\bar{\theta}}{2M}\right)d\rho \quad (24)$$

Another form to the above equations:

$$E^{(0)}_{\text{u-}dwao}=|N_0|^2\gamma_+\left(j=l+1/2,l,s=1/2\right)\left\{\theta\sum_{\mu=1}^{2}I^{s}_{0-\mu}+\frac{\bar{\theta}}{2M}I^{p}_{0-3}\right\}$$

$$E^{(0)}_{\text{d-}dwao}=|N_0|^2\gamma_-\left(j=l+1/2,l,s=1/2\right)\left\{\theta\sum_{\mu=1}^{2}I^{s}_{0-\mu}+\frac{\bar{\theta}}{2M}I^{p}_{0-3}\right\} \quad (25)$$

Where, the three terms $I^{s}_{0-\mu}\left(i=\overline{1,2}\right)$ and I^{p}_{0-3} are given by:

$$I^{s}_{0-1}=\frac{1}{2}M\omega^2I^{p}_{0-3}=\frac{1}{2}M\omega^2\int_0^{+\infty}\rho^{2|m|+1}\exp\left(-\frac{\rho^4}{4}\right)d\rho, \quad (26)$$

$$I^{s}_{0-2}=-\frac{\lambda}{4}\int_0^{+\infty}\rho^{2|m|+5}\exp\left(-\frac{\rho^4}{4}\right)d\rho$$

It is convenient to using the following special integral [46]:

$$\int_0^{+\infty}x^{\nu-1}\exp\left(-\mu x^{p}\right)dx=\frac{1}{p}\mu^{-\frac{\nu}{p}}\Gamma\left(\frac{\nu}{p}\right) \quad (27)$$

With conditions ($\mathrm{Re}\,\mu\rangle 0$, $\mathrm{Re}\,\nu\rangle 0$ and $p\rangle 0$) and $\Gamma\left(\frac{\nu}{p}\right)$ the ordinary Gamma function.

After straightforward calculations, we can obtain the explicitly results:

$$I^{s}_{0-1}=\frac{1}{2}M\omega^2I^{p}_{0-3}=\frac{2M\omega^2}{4^{8|m|+8}}\Gamma\left(8|m|+8\right)$$

$$I^{s}_{0-2}=-\frac{\lambda}{4^{8|m|+24}}\Gamma\left(8|m|+24\right) \quad (28)$$

Which allow us to obtaining the exact modifications ($E^{(0)}_{\text{u-}dwao}$ and $E^{(0)}_{\text{d-}dwao}$) produced by spin-orbital effect:

$$E^{(0)}_{\text{u-}dwao}=|N_0|^2\gamma_+\left(j=l+1/2,l,s=1/2\right)\left\{\theta I^{s}_{0-dwao}+\frac{\bar{\theta}}{2M}I^{p}_{0-dwao}\right\}$$

$$E^{(0)}_{\text{d-}dwao}=|N_0|^2\gamma_-\left(j=l+1/2,l,s=1/2\right)\left\{\theta I^{s}_{0-dwao}+\frac{\bar{\theta}}{2M}I^{p}_{0-dwao}\right\} \quad (29)$$

we have introduced new coefficients I^{s}_{0-dwao} and I^{p}_{0-dwao} for the sake of simplicity:

$$I^{s}_{0-dwao}\equiv\frac{1}{4^{8|m|+8}}\left(2M\omega^2\Gamma\left(8|m|+8\right)-\frac{\lambda}{4^{16}}\Gamma\left(8|m|+24\right)\right) \text{ and } I^{p}_{0-dwao}=\frac{4}{4^{8|m|+7}}\Gamma\left(8|m|+8\right) \quad (30)$$

4.1.2 STUDY THE CASE $N=1$ AND GEOMETRY VALUES $\beta=4+|m|$ WITH PRINCIPAL QUANTUM NUMBER $n=1+|m|$

By applying the two axioms related to the energy

$$E^{(1)}_{\text{u-}dwao}=\left\langle \Psi_{n=1+|m|,m}(\rho,\varphi)\left(H^{so}_{dwao}\right)_{11}\Psi_{n=1+|m|,m}(\rho,\varphi)\right\rangle$$

$$E^{(1)}_{\text{d-}dwao}=\left\langle \Psi_{n=1+|m|,m}(\rho,\varphi)\left(H^{so}_{dwao}\right)_{22}\Psi_{n=1+|m|,m}(\rho,\varphi)\right\rangle$$

and using the wave equation in eq. (11), we may obtain the modifications to the energy levels for 2D-modified sextic DWAO potential $\left(E^{(0)}_{\text{u-}dwao},E^{(0)}_{\text{d-}dwao}\right)$ for the case $N=1$ and geometry values $\beta=4+|m|$ with principal quantum number $n=1+|m|$:

$$E^{(1)}_{\text{u-}dwao}\left(n=1+|m|\right)=|N_1|^2\gamma_+\int_0^{+\infty}\rho^{2|m|+1}\exp\left(-\frac{\rho^4}{4}\right)\left(1+\frac{2\rho^2}{\sqrt{4|m|+2}}+\frac{\rho^4}{4|m|+2}\right)\left(\theta\left(\frac{1}{2}M\omega^2-\frac{\lambda}{4}\rho^4\right)+\frac{\bar{\theta}}{2M}\right)d\rho$$

$$E^{(1)}_{\text{d-}dwao}\left(n=1+|m|\right)=|N_1|^2\gamma_-\int_0^{+\infty}\rho^{2|m|+1}\exp\left(-\frac{\rho^4}{4}\right)\left(1+\frac{2\rho^2}{\sqrt{4|m|+2}}+\frac{\rho^4}{4|m|+2}\right)\left(\theta\left(\frac{1}{2}M\omega^2-\frac{\lambda}{4}\rho^4\right)+\frac{\bar{\theta}}{2M}\right)d\rho \quad (31)$$

This can be rewritten as:

$$E^{(1)}_{\text{u-}dwao}\left(n=1+|m|\right)=|N_1|^2\gamma k_+\left(j=l+1/2,l,s=1/2\right)\left\{\theta\sum_{\mu=1}^{6}I^{s}_{1-\mu}+\frac{\bar{\theta}}{2M}\sum_{\nu=7}^{9}I^{p}_{1-\nu}\right\}$$
$$E^{(1)}_{\text{d-}dwao}\left(n=1+|m|\right)=|N_1|^2\gamma k_-\left(j=l+1/2,l,s=1/2\right)\left\{\theta\sum_{\mu=1}^{6}I^{s}_{1-\mu}+\frac{\bar{\theta}}{2M}\sum_{\nu=7}^{9}I^{p}_{1-\nu}\right\} \quad (32)$$

Where, the 9-terms $I^{s}_{1-\mu}\left(\mu=\overline{1,6}\right)$ and $I^{p}_{1-\nu}\left(\nu=\overline{7,9}\right)$ are given by:

$$I^{s}_{1-1}=\frac{1}{2}M\omega^2\int_0^{+\infty}\rho^{2|m|+2-1}\exp\left(-\frac{\rho^4}{4}\right)d\rho,$$
$$I^{s}_{1-2}=\frac{M\omega^2}{\sqrt{4|m|+2}}\int_0^{+\infty}\rho^{2|m|+4-1}\exp\left(-\frac{\rho^4}{4}\right)d\rho,$$
$$I^{s}_{1-3}=\frac{M\omega^2}{8|m|+4}\int_0^{+\infty}\rho^{2|m|+6-1}\exp\left(-\frac{\rho^4}{4}\right)d\rho \quad (33)$$
$$I^{s}_{1-4}=-\frac{\lambda}{4}\int_0^{+\infty}\rho^{2|m|+6-1}\exp\left(-\frac{\rho^4}{4}\right)d\rho,$$
$$I^{s}_{1-5}=-\frac{2\lambda}{\sqrt{8|m|+4}}\int_0^{+\infty}\rho^{2|m|+8-1}\exp\left(-\frac{\rho^4}{4}\right)d\rho,$$
$$I^{s}_{1-6}=-\frac{\lambda}{16|m|+8}\int_0^{+\infty}\rho^{2|m|+10-1}\exp\left(-\frac{\rho^4}{4}\right)d\rho \quad (34)$$

$$I^{p}_{1-7}=\int_0^{+\infty}\rho^{2|m|+2-1}\exp\left(-\frac{\rho^4}{4}\right)d\rho,$$
$$I^{p}_{1-8}=\frac{2}{\sqrt{4|m|+2}}\int_0^{+\infty}\rho^{2|m|+4-1}\exp\left(-\frac{\rho^4}{4}\right)d\rho,$$
$$I^{p}_{1-9}=\frac{1}{4|m|+2}\int_0^{+\infty}\rho^{2|m|+6-1}\exp\left(-\frac{\rho^4}{4}\right)d\rho \quad (35)$$

It's convenient to using the previously special integral, we can obtain the explicitly results:

$$I^{s}_{1-1}=\frac{1}{8}M\omega^2 4^{\frac{|m|+1}{2}}\Gamma\left(\frac{|m|+1}{2}\right),$$
$$I^{s}_{1-2}=\frac{M\omega^2}{4\sqrt{4|m|+2}}4^{\frac{|m|+2}{2}}\Gamma\left(\frac{|m|+2}{2}\right) \quad (36)$$
$$I^{s}_{1-3}=\frac{M\omega^2}{32|m|+16}4^{\frac{|m|+3}{2}}\Gamma\left(\frac{|m|+3}{2}\right)$$

$$I^{s}_{1-4}=-\frac{\lambda}{16}4^{\frac{|m|+3}{2}}\Gamma\left(\frac{|m|+3}{2}\right),$$
$$I^{s}_{1-5}=-\frac{2\lambda}{4\sqrt{8|m|+4}}4^{\frac{|m|+4}{2}}\Gamma\left(\frac{|m|+4}{2}\right), \quad (37)$$
$$I^{s}_{1-6}=-\frac{\lambda}{64|m|+32}4^{\frac{|m|+5}{2}}\Gamma\left(\frac{|m|+5}{2}\right)$$

$$I^{p}_{1-7}=\frac{1}{4}4^{\frac{|m|+2}{2}}\Gamma\left(\frac{|m|+2}{2}\right),$$
$$I^{p}_{1-8}=\frac{1}{\sqrt{16|m|+8}}4^{\frac{|m|+2}{2}}\Gamma\left(\frac{|m|+2}{2}\right) \quad (38)$$
$$I^{p}_{1-9}=\frac{1}{16|m|+8}4^{\frac{|m|+3}{2}}\Gamma\left(\frac{|m|+3}{2}\right)$$

Which allow us to obtaining the exact modifications ($E^{(1)}_{\text{u-}dwao}\left(n=1+|m|\right)$ and $E^{(1)}_{\text{d-}dwao}\left(n=1+|m|\right)$) produced by spin-orbital effect:

$$E^{(1)}_{\text{u-}dwao}\left(n=1+|m|\right)=|N_1|^2\gamma k_+\left(j=l+1/2,l,s=1/2\right)\left\{\theta I^{s}_{1-dwao}+\frac{\bar{\theta}}{2M}I^{p}_{1-dwao}\right\}$$
$$E^{(1)}_{\text{d-}dwao}\left(n=1+|m|\right)=|N_0|^2\gamma k_-\left(j=l+1/2,l,s=1/2\right)\left\{\theta I^{s}_{1-dwao}+\frac{\bar{\theta}}{2M}I^{p}_{1-dwao}\right\} \quad (39)$$

we have introduced new coefficients I^{s}_{1-dwao} and I^{p}_{1-dwao} for the sake of simplicity:

$$I^{s}_{1-dwao}\equiv\sum_{\mu=1}^{6}I^{s}_{1-\mu} \quad \text{and} \quad I^{p}_{1-dwao}=\frac{\bar{\theta}}{2M}\sum_{\nu=7}^{9}I^{p}_{1-\nu} \quad (40)$$

4.1.3 STUDY THE CASE $N=1$ AND GEOMETRY VALUES $\beta=4+|m|$ WITH PRINCIPAL QUANTUM NUMBER $n=2+|m|$

By applying:

$$E^{(1)}_{\text{u-}dwao}=\left\langle\Psi_{n=1+|m|,m}(\rho,\varphi)\middle|\left(H^{so}_{dwao}\right)_{11}\middle|\Psi_{n=1+|m|,m}(\rho,\varphi)\right\rangle$$

and

$$E^{(1)}_{\text{d-}dwao}=\left\langle\Psi_{n=1+|m|,m}(\rho,\varphi)\middle|\left(H^{so}_{dwao}\right)_{22}\middle|\Psi_{n=1+|m|,m}(\rho,\varphi)\right\rangle$$

using the wave equation in eq. (10), we may obtain the modifications to the energy levels for 2D-modified sextic DWAO

potential $\left(E^{(0)}_{\text{u-}dwao}, E^{(0)}_{\text{d-}dwao}\right)$ for the case $N=1$ and geometry values $\beta = 4+|m|$ with principal quantum number $n = 2+|m|$:

$$E^{(1)}_{\text{u-}dwao}\left(n=2+|m|\right)=|N_1|^2\gamma k_+\int_0^{+\infty}\rho^{2|m|+1}\exp\left(-\frac{\rho^4}{4}\right)\left(1-\frac{2\rho^2}{\sqrt{4|m|+2}}+\frac{\rho^4}{4|m|+2}\right)\left(\theta\left(\frac{1}{2}M\omega^2-\frac{\lambda}{4}\rho^4\right)+\frac{\bar{\theta}}{2M}\right)d\rho$$

$$E^{(1)}_{\text{d-}dwao}\left(n=2+|m|\right)=|N_1|^2\gamma k_-\int_0^{+\infty}\rho^{2|m|+1}\exp\left(-\frac{\rho^4}{4}\right)\left(1-\frac{2\rho^2}{\sqrt{4|m|+2}}+\frac{\rho^4}{4|m|+2}\right)\left(\theta\left(\frac{1}{2}M\omega^2-\frac{\lambda}{4}\rho^4\right)+\frac{\bar{\theta}}{2M}\right)d\rho \quad (41)$$

An equivalent new form:

$$E^{(1)}_{\text{u-}dwao}\left(n=2+|m|\right)=|N_1|^2\gamma k_+\left(j=l+1/2,l,s=1/2\right)\left\{\theta\sum_{\mu=1}^{6}T^s_{1-\mu}+\frac{\bar{\theta}}{2M}\sum_{\nu=7}^{9}T^p_{1-\nu}\right\}$$

$$E^{(1)}_{\text{d-}dwao}\left(n=2+|m|\right)=|N_1|^2\gamma k_-\left(j=l+1/2,l,s=1/2\right)\left\{\theta\sum_{\mu=1}^{6}T^s_{1-\mu}+\frac{\bar{\theta}}{2M}\sum_{\nu=7}^{9}T^p_{1-\nu}\right\} \quad (42)$$

Where, the 9-terms $T^s_{1-\mu}\left(\mu=\overline{1,6}\right)$ and $T^p_{1-\nu}\left(\nu=\overline{7,9}\right)$ are given by:

$$T^s_{1-1}=\frac{1}{2}M\omega^2\int_0^{+\infty}\rho^{2|m|+2-1}\exp\left(-\frac{\rho^4}{4}\right)d\rho,$$

$$T^s_{1-2}=-\frac{M\omega^2}{\sqrt{4|m|+2}}\int_0^{+\infty}\rho^{2|m|+4-1}\exp\left(-\frac{\rho^4}{4}\right)d\rho, \quad (43)$$

$$T^s_{1-3}=\frac{M\omega^2}{8|m|+4}\int_0^{+\infty}\rho^{2|m|+6-1}\exp\left(-\frac{\rho^4}{4}\right)d\rho$$

$$T^s_{1-4}=-\frac{\lambda}{4}\int_0^{+\infty}\rho^{2|m|+6-1}\exp\left(-\frac{\rho^4}{4}\right)d\rho,$$

$$T^s_{1-5}=\frac{2\lambda}{\sqrt{8|m|+4}}\int_0^{+\infty}\rho^{2|m|+8-1}\exp\left(-\frac{\rho^4}{4}\right)d\rho, \quad (44)$$

$$T^s_{1-6}=-\frac{\lambda}{16|m|+8}\int_0^{+\infty}\rho^{2|m|+10-1}\exp\left(-\frac{\rho^4}{4}\right)d\rho$$

$$T^p_{1-7}=\int_0^{+\infty}\rho^{2|m|+2-1}\exp\left(-\frac{\rho^4}{4}\right)d\rho,$$

$$T^p_{1-8}=-\frac{2}{\sqrt{4|m|+2}}\int_0^{+\infty}\rho^{2|m|+4-1}\exp\left(-\frac{\rho^4}{4}\right)d\rho, \quad (45)$$

$$T^p_{1-9}=\frac{1}{4|m|+2}\int_0^{+\infty}\rho^{2|m|+6-1}\exp\left(-\frac{\rho^4}{4}\right)d\rho$$

It is convenient to using the previously special integral, we can obtain the explicitly results:

$$T^s_{1-1}=\frac{1}{8}M\omega^2 4^{\frac{|m|+1}{2}}\Gamma\left(\frac{|m|+1}{2}\right),$$

$$T^s_{1-2}=-\frac{M\omega^2}{4\sqrt{4|m|+2}}4^{\frac{|m|+2}{2}}\Gamma\left(\frac{|m|+2}{2}\right) \quad (46)$$

$$T^s_{1-3}=\frac{M\omega^2}{32|m|+16}4^{\frac{|m|+3}{2}}\Gamma\left(\frac{|m|+3}{2}\right)$$

$$T^s_{1-4}=-\frac{\lambda}{16}4^{\frac{|m|+3}{2}}\Gamma\left(\frac{|m|+3}{2}\right),$$

$$T^s_{1-5}=\frac{2\lambda}{4\sqrt{8|m|+4}}4^{\frac{|m|+4}{2}}\Gamma\left(\frac{|m|+4}{2}\right), \quad (47)$$

$$T^s_{1-6}=-\frac{\lambda}{64|m|+32}4^{\frac{|m|+5}{2}}\Gamma\left(\frac{|m|+5}{2}\right)$$

$$T^p_{1-7}=\frac{1}{4}4^{\frac{|m|+2}{2}}\Gamma\left(\frac{|m|+2}{2}\right),$$

$$T^p_{1-8}=-\frac{1}{\sqrt{16|m|+8}}4^{\frac{|m|+2}{2}}\Gamma\left(\frac{|m|+2}{2}\right), \quad (48)$$

$$T^p_{1-9}=\frac{1}{16|m|+8}4^{\frac{|m|+3}{2}}\Gamma\left(\frac{|m|+3}{2}\right)$$

Which allow us to obtaining the exact modifications ($E^{(1)}_{\text{u-}dwao}\left(n=2+|m|\right)$ and $E^{(1)}_{\text{d-}dwao}\left(n=2+|m|\right)$) produced by spin-orbital effect:

$$E^{(1)}_{\text{u-}dwao}\left(n=2+|m|\right)=|N_1|^2\gamma k_+\left(j=l+1/2,l,s=1/2\right)\left\{\theta T^s_{1-dwao}+\frac{\bar{\theta}}{2M}T^p_{1-dwao}\right\}$$

$$E^{(1)}_{\text{d-}dwao}\left(n=2+|m|\right)=|N_1|^2\gamma k_-\left(j=l+1/2,l,s=1/2\right)\left\{\theta T^s_{1-dwao}+\frac{\bar{\theta}}{2M}T^p_{1-dwao}\right\} \quad (49)$$

we have introduced new coefficients T^s_{1-dwao} and T^p_{1-dwao} for the sake of simplicity:

$$T^s_{1-dwao}\equiv\sum_{\mu=1}^{6}T^s_{1-\mu} \text{ and } T^p_{1-dwao}=\frac{\bar{\theta}}{2M}\sum_{\nu=7}^{9}T^p_{1-\nu} \quad (50)$$

5. THE EXACT MAGNETIC SPECTRUM FOR 2D-MODIFIED SEXTIC DWAO POTENTIAL IN (NC:2D- RSP)

On the other hand, we consider interested second physically meaningful phenomena, which produced automatically from the perturbative terms of 2D-modified sextic DWAO potential related to the influence of an external uniform magnetic field, it is sufficient to apply the following replacements to describing these phenomena:

$$\begin{cases}\theta \to \chi B \\ \bar{\theta} \to \bar{\sigma} B\end{cases} \Rightarrow \left(\theta\left(\frac{1}{2}M\omega^2 - \frac{\lambda}{4}\rho^4\right) + \frac{\bar{\theta}}{2M}\right)L_z \to \left(\chi\left(\frac{1}{2}M\omega^2 - \frac{\lambda}{4}\rho^4\right) + \frac{\bar{\sigma}}{2M}\right)\vec{B}\vec{L} \tag{51}$$

here χ and $\bar{\sigma}$ are infinitesimal real proportional's constants, and we choose the magnetic field $\vec{B} = \mathbf{B}\vec{k}$, which allow us to introduce the modified new magnetic Hamiltonian H_m^{dwao} in (NC: 2D-RSP) as:

$$H_m^{dwao} = \left\{\chi\left(\frac{1}{2}M\omega^2 - \frac{\lambda}{4}\rho^4\right) + \frac{\bar{\sigma}}{2M}\right\}\left(\vec{B}\vec{J} - \vec{B}\vec{S}\right) \tag{52}$$

where $\vec{B}\vec{S}$ denote to the ordinary Hamiltonian of Zeeman Effect. To obtain the exact NC magnetic modifications of energy ($E_{\text{m-}dwao}^{(N)}(n)$) for 2D-modified sextic DWAO potential corresponding the case $N=0$ and geometry values $\beta = 2+|m|$ with principal quantum number $n = 1+|m|$, the case $N=1$ and geometry values $\beta = 4+|m|$ with principal quantum number $n = 1+|m|$ and the case $N=1$ and geometry values $\beta = 4+|m|$ with principal quantum number $n = 2+|m|$, respectively, we make the three translations:

$$k_+ \to \mathrm{m}, \theta \to \chi \quad \text{and} \quad \bar{\theta} \to \bar{\sigma} \tag{53}$$

we insert the values of the parameters given in eq. (53) into eqs. (29), (39) and (49) to obtain ($E_{\text{m-}dwao}^{(0)}(n = 1+|m|)$, $E_{\text{m-}dwao}^{(1)}(n = 1+|m|)$ and $E_{\text{m-}dwao}^{(1)}(n = 2+|m|)$) as follows:

$$\begin{aligned}
E_{\text{m-}dwao}^{(0)}(n = 1+|m|) &= |N_0|^2 \gamma\left\{\chi I_{0-dwao}^{s} + \frac{\bar{\sigma}}{2M} I_{0-dwao}^{p}\right\}Bm \\
E_{\text{m-}dwao}^{(1)}(n = 1+|m|) &= |N_1|^2 \gamma\left\{\chi I_{1-dwao}^{s} + \frac{\bar{\sigma}}{2M} I_{1-dwao}^{p}\right\}Bm \\
E_{\text{m-}dwao}^{(1)}(n = 2+|m|) &= |N_1|^2 \gamma\left\{\chi T_{1-dwao}^{s} + \frac{\bar{\sigma}}{2M} T_{1-dwao}^{p}\right\}Bm
\end{aligned} \tag{54}$$

we have $-l \le m \le +l$, which allow us to fixing ($2l+1$) values for the orbital angular momentum quantum numbers.

6. RESULTS

Let us now resume the eigenenergies of the DSE obtained in this paper, the total modified energies ($E_{nc-udwao}^{(N)}(n)$ and $E_{nc-udwao}^{(N)}(n)$) of a particle fermionic with spin up and spin down are determined corresponding fundamental states and first excited states, respectively, for 2D-modified sextic DWAO potential in (NC: 2D-RSP), on based to the eqs. (9), (10), (29), (29), (49), and (54), we obtain the four new values of global energies:

The case $N=0$ and geometry values $\beta = 2+|m|$ with principal quantum number $n = 1+|m|$

$$\begin{aligned}
E_{nc-udwao}^{(0)}(n = 1+|m|) &= |N_0|^2 \gamma k_+\left\{\theta I_{0-dwao}^{s} + \frac{\bar{\theta}}{2M} I_{0-dwao}^{p}\right\} + |N_0|^2 \gamma\left\{\chi I_{0-dwao}^{s} + \frac{\bar{\sigma}}{2M} I_{0-dwao}^{p}\right\}Bm \\
E_{nc-ddwao}^{(0)}(n = 1+|m|) &= |N_0|^2 \gamma k_-\left\{\theta I_{0-dwao}^{s} + \frac{\bar{\theta}}{2M} I_{0-dwao}^{p}\right\} + |N_0|^2 \gamma\left\{\chi I_{0-dwao}^{s} + \frac{\bar{\sigma}}{2M} I_{0-dwao}^{p}\right\}Bm
\end{aligned} \tag{55}$$

The case $N=1$ and geometry values $\beta = 4+|m|$ with principal quantum number $n = 1+|m|$

$$E^{(1)}_{nc-udwao}\left(n=1+|m|\right)=-\sqrt{2|m|+2}+|N_1|^2\gamma k_+\left\{\theta I^s_{1-dwao}+\frac{\bar{\theta}}{2M}I^p_{1-dwao}\right\}+|N_1|^2\gamma\left\{\chi I^s_{1-dwao}+\frac{\bar{\sigma}}{2M}I^p_{1-dwao}\right\}Bm$$
$$E^{(1)}_{nc-ddwao}\left(n=1+|m|\right)=-\sqrt{2|m|+2}+|N_1|^2\gamma k_-\left\{\theta I^s_{1-dwao}+\frac{\bar{\theta}}{2M}I^p_{1-dwao}\right\}+|N_1|^2\gamma\left\{\chi I^s_{1-dwao}+\frac{\bar{\sigma}}{2M}I^p_{1-dwao}\right\}Bm \quad (56)$$

The case $N=1$ and geometry values $\beta=4+|m|$ with principal quantum number $n=2+|m|$ It's clearly, that the obtained eigenvalues of energies are real and then the NC diagonal Hamiltonian H^{dwao}_{nc} is Hermitian for 2D-modified sextic DWAO potential, furthermore it's possible to writing the non-null elements $\left(H^{dwao}_{nc}\right)_{11}$ and $\left(H^{dwao}_{nc}\right)_{22}$ as follows:

$$\left(H^{dwao}_{nc}\right)_{11}=-\frac{\Delta_{nc}}{2M}+H^{udwao}_{\text{int}}$$
$$\left(H^{dwao}_{nc}\right)_{22}=-\frac{\Delta_{nc}}{2M}+H^{ddwao}_{\text{int}} \quad (57)$$

Where the new kinetic energy is $\frac{\Delta_{nc}}{2M}$ while the two modified interactions H^{udwao}_{int} and H^{ddwao}_{int} for 2D-modified sextic DWAO potential are given by:

$$\frac{\Delta_{nc}}{2M}=\frac{\Delta-\bar{\theta}L_z-\bar{\sigma}L_z}{2M}$$
$$H^{udwao}_{\text{int}}=-\frac{1}{2}M\omega^2\rho^2+\frac{\lambda}{8}\rho^6+\left(H^{so}_{dwao}\right)_{11}+H^{dwao}_{m} \quad (58)$$
$$H^{ddwao}_{\text{int}}=-\frac{1}{2}M\omega^2\rho^2+\frac{\lambda}{8}\rho^6+\left(H^{so}_{dwao}\right)_{22}+H^{dwao}_{m}$$

Thus, the ordinary kinetic term for 2D-sexticDWAO potential $\left(-\frac{\Delta}{2M}\right)$ and ordinary interaction $(-\frac{1}{2}M\omega^2\rho^2+\frac{\lambda}{8}\rho^6)$ are replaced by new modified form of kinetic term $\left(-\frac{\Delta_{nc}}{2M}\right)$ and new modified interactions (H^{udwao}_{int} and H^{ddwao}_{int}). Which allows us to obtain the original results for this investigation: the obtained Hamiltonian operator (69) describing atom, which has two permanent dipoles: the first is electric dipole moment and the second is magnetic moment in external stationary electromagnetic field. Regarding to the wave functions for the case $N=2$ and geometry values $\beta=6+|m|$ with three principal quantum numbers $n=3+|m|$, $n=2+|m|$ and $n=1+|m|$, we can easily generalized our previously results to include the case $N=2$ and geometry values $\beta=6+|m|$ as follows: For the principal quantum number $n=3+|m|$ as follows:

$$E^{(2)}_{nc-udwao}\left(n=3+|m|\right)=2\sqrt{2|m|+2}+|N_2|^2\gamma k_+\left\{\theta\sum_{\nu=1}^{12}T^s_{\nu-2dwao}+\frac{\bar{\theta}}{2M}\sum_{\nu=13}^{18}T^p_{\nu-2dwao}\right\}+|N_1|^2\gamma\left\{\chi\sum_{\nu=1}^{12}T^s_{\nu-2dwao}+\frac{\bar{\sigma}}{2M}\sum_{\nu=13}^{18}T^p_{\nu-2dwao}\right\}Bm$$
$$E^{(2)}_{nc-ddwao}\left(n=3+|m|\right)=2\sqrt{2|m|+2}+|N_2|^2\gamma k_-\left\{\theta\sum_{\nu=1}^{12}T^s_{\nu-2dwao}+\frac{\bar{\theta}}{2M}\sum_{\nu=13}^{18}T^p_{\nu-2dwao}\right\}+|N_1|^2\gamma\left\{\chi\sum_{\nu=1}^{12}T^s_{\nu-2dwao}+\frac{\bar{\sigma}}{2M}\sum_{\nu=13}^{18}T^p_{\nu-2dwao}\right\}Bm \quad (59)$$

For the principal quantum number $n=2+|m|$, we have:

$$E^{(2)}_{nc-udwao}\left(n=2+|m|\right)=|N_2|^2\gamma k_+\left\{\theta\sum_{\nu=1}^{6}T^s_{\nu-2dwao}+\frac{\bar{\theta}}{2M}\sum_{\nu=7}^{9}T^p_{\nu-2dwao}\right\}+|N_1|^2\gamma\left\{\chi\sum_{\nu=1}^{6}T^s_{\nu-2dwao}+\frac{\bar{\sigma}}{2M}\sum_{\nu=7}^{9}T^p_{\nu-2dwao}\right\}Bm$$
$$E^{(2)}_{nc-ddwao}\left(n=2+|m|\right)=|N_2|^2\gamma k_-\left\{\theta\sum_{\nu=1}^{6}T^s_{\nu-2dwao}+\frac{\bar{\theta}}{2M}\sum_{\nu=7}^{9}T^p_{\nu-2dwao}\right\}+|N_1|^2\gamma\left\{\chi\sum_{\nu=1}^{6}T^s_{\nu-2dwao}+\frac{\bar{\sigma}}{2M}\sum_{\nu=7}^{9}T^p_{\nu-2dwao}\right\}Bm \quad (60)$$

For the principal quantum number $n=1+|m|$ we have:

$$E^{(2)}_{nc-udwao}\left(n=1+|m|\right)=-2\sqrt{2|m|+2}+|N_2|^2\gamma k_+\left\{\theta\sum_{\nu=1}^{12}T^s_{\nu-2dwao}+\frac{\bar{\theta}}{2M}\sum_{\nu=13}^{18}T^p_{\nu-2dwao}\right\}+|N_1|^2\gamma\left\{\chi\sum_{\nu=1}^{12}T^s_{\nu-2dwao}+\frac{\bar{\sigma}}{2M}\sum_{\nu=13}^{18}T^p_{\nu-2dwao}\right\}Bm$$
$$E^{(2)}_{nc-ddwao}\left(n=1+|m|\right)=-2\sqrt{2|m|+2}+|N_2|^2\gamma k_-\left\{\theta\sum_{\nu=1}^{12}T^s_{\nu-2dwao}+\frac{\bar{\theta}}{2M}\sum_{\nu=13}^{18}T^p_{\nu-2dwao}\right\}+|N_1|^2\gamma\left\{\chi\sum_{\nu=1}^{12}T^s_{\nu-2dwao}+\frac{\bar{\sigma}}{2M}\sum_{\nu=13}^{18}T^p_{\nu-2dwao}\right\}Bm \quad (61)$$

Now, we return to the degeneracy of energy level of 2D-modified sextic DWAO potential, we need to sum for all allowed values of l. Total degeneracy is thus,

$$2\sum_{i=0}^{n-1}(2l+1)\equiv 2n^2 \quad \text{for the case N = 0 with one geometry values}$$
$$2\left(2\sum_{i=0}^{n-1}(2l+1)\right)\equiv 4n^2 \text{ for the case N = 1 with two geometry values}$$
$$3\left(2\sum_{i=0}^{n-1}(2l+1)\right)\equiv 6n^2 \text{ for the case N = 2 with three geometry values} \quad (62)$$

7. GENERALIZATION THE OBTAINED RESULTS TO THE ENERGY SPECTRA FOR THE ATOMS FOR ($\vec{S}\neq 1/\vec{2}$)

As it is mentioned in the introduction, the 2D-modified sextic DWAO potential can be described the interactions in atoms with spin $\vec{S}\neq 1/\vec{2}$ in (NC: 2D-RSP), the atomic quantum number m and j can be takes ($2l+1$) values and $\underbrace{j=|l-s|,|l-s|+1,...,...j=|l+s| \quad \text{and} \quad j=|l+s|}_{\text{N-possible values for j}}$ values, respectively, thus every old state in usually two dimensional space will be $N(2l+1)$sub-states for 2D-modified sextic DWAO potential. Furthermore, we can deduce the global noncommutative Hamiltonian $H_{nc-dwao}$ in both (NC_2D: RSP) for studied potential on based to the previously obtained results:

$$H_{nc-dwao}=-\frac{1}{2M}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right)-\frac{1}{2}M\omega^2\rho^2+\frac{\lambda}{8}\rho^6+\gamma\left(\theta\left(\frac{1}{2}M\omega^2-\frac{\lambda}{4}\rho^4\right)+\frac{\bar{\theta}}{2M}\right)\vec{L}\vec{S}+$$
$$+\left\{\chi\left(\frac{1}{2}M\omega^2-\frac{\lambda}{4}\rho^4\right)+\frac{\bar{\sigma}}{2M}\right\}\left(\vec{B}\vec{J}-\vec{B}\vec{S}\right) \quad (63)$$

the corresponding energy spectra for three special cases: The case $N=0$ and geometry values $\beta=2+|m|$ with principal quantum number $n=1+|m|$

$$E^{(0)}_{nc-udwao}\left(n=1+|m|\right)=|N_0|^2\gamma k(j,l,s)\left\{\theta I^{s}_{0-dwao}+\frac{\bar{\theta}}{2M}I^{p}_{0-dwao}\right\}+|N_0|^2\gamma\left\{\chi I^{s}_{0-dwao}+\frac{\bar{\sigma}}{2M}I^{p}_{0-dwao}\right\}Bm \quad (64)$$

The case $N=1$ and geometry values $\beta=4+|m|$ with principal quantum number $n=1+|m|$

$$E^{(1)}_{nc-udwao}\left(n=1+|m|\right)=-\sqrt{2|m|+2}+|N_1|^2\gamma k(j,l,s)\left\{\theta I^{s}_{1-dwao}+\frac{\bar{\theta}}{2M}I^{p}_{1-dwao}\right\}+|N_1|^2\gamma\left\{\chi I^{s}_{1-dwao}+\frac{\bar{\sigma}}{2M}I^{p}_{1-dwao}\right\}Bm \quad (65)$$

The case $N=1$ and geometry values $\beta=4+|m|$ with principal quantum number $n=2+|m|$

$$E^{(1)}_{nc-udwao}\left(n=2+|m|\right)=\sqrt{2|m|+2}+|N_1|^2\gamma k(j,l,s)\left\{\theta T^{s}_{1-dwao}+\frac{\bar{\theta}}{2M}T^{p}_{1-dwao}\right\}+|N_1|^2\gamma\left\{\chi T^{s}_{1-dwao}+\frac{\bar{\sigma}}{2M}T^{p}_{1-dwao}\right\}Bm \quad (66)$$

We have replaced the factor $k_{\pm}\left(j=l\pm 1/2,l,s=1/2\right)$ by new values $k(j,l,s)\equiv\frac{1}{2}\left\{j(j+1)+l(l+1)-s(s+1)\right\}$. Paying attention to the behaviour of the spectrums: (55), (56), (57), (59), (60), (61), (64), (65) and (66), it is possible to recover the results of commutative space when we consider $(\theta,\chi)\to(0,0)$.

8. CONCLUSION

To summarize, this work has been devoted to find the solutions of 2D-modified sextic DWAO potential, we have obtained the exact energy spectrum for three special cases: the case $N=0$ and geometry values $\beta=2+|m|$ with principal quantum number $n=1+|m|$, the case $N=1$ and geometry values $\beta=4+|m|$ with principal quantum number $n=1+|m|$ and the case $N=1$ and geometry values $\beta=4+|m|$ with principal quantum number $n=2+|m|$ in (NC: 2D-RSP) symmetries. We shown that the old states were changed radically and replaced by degenerated new states, describing two new original spectrums, the first one, produced by the of spin-orbital interaction while the second new spectrum produced by an external magnetic field, determined by the new results (55), (56), (57), (59), (60) and (61). Furthermore, our recently study can be generalized to

including the interactions for atoms with spin $\vec{S} \neq \vec{1/2}$ which describing by the new spectrums (64), (65) and (66).

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