Yanbu Journal of Engineering and Science

Volume 16 | Issue 1 Article 6

2018

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Recommended Citation
Assabaai, M. A. (2018), NUMERICAL SOLUTION OF CONVECTION-DIFFUSION EQUATION BY CHEBYSHEV SPECTRAL METHOD VIA LIE GROUP METHOD, *Yanbu Journal of Engineering and Science*: Vol. 16: Iss. 1, 57-64.
DOI: https://doi.org/10.53370/001c.24343
Available at: https://yjes.researchcommons.org/yjes/vol16/iss1/6

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Yanbu Journal
of
Engineering
and Science
ISSN: 1658-5321

Vol. 16, June 2018 (1439H)
www.yjes.org.sa

NUMERICAL SOLUTION OF CONVECTION-DIFFUSION EQUATION BY CHEBYSHEV SPECTRAL METHOD VIA LIE GROUP METHOD

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ABSTRACT

The numerical solution of convection-diffusion equation is presented by using Chebyshev spectral method based on El-Gendi method via Lie group analysis. Firstly, we apply Lie symmetry group analysis for the convection-diffusion equation. This method yields convection-diffusion equation to a system of ordinary differential equations (ODEs). Secondly, this system is solved numerically by using Chebyshev spectral method. The numerical results obtained by this way are compared with the exact solution.

Keywords: *Chebyshev spectral method, Convection-diffusion equation, Lie group analysis.*

1. INTRODUCTION

In this paper, we will consider the convection-diffusion equation;

$$\frac{\partial u}{\partial t}+\alpha\frac{\partial u}{\partial x}-\varepsilon\frac{\partial^2 u}{\partial x^2}=0,\ 0\le x\le 1,\ 0\le t\le T \qquad (1)$$

with initial condition

$$u(x,0)=f(x),\quad 0\le x\le 1 \qquad (2)$$

and perturbed boundary conditions

$$\begin{aligned} u(0,t)&=g_0(t)+\varepsilon_0,\quad 0\le t\le T,\\ u(1,t)&=g_1(t)+\varepsilon_1,\quad 0\le t\le T \end{aligned} \qquad (3)$$

Where f, g_0 and g_1 are known functions, the parameters α and ε are convection and diffusion coefficients respectively and both are assumed to be positive, and ε_0 , ε_1 are small positive numbers**.** A large number of problems in physics, chemistry, financial mathematics and many other fields can be modelled by convection-diffusion equation, (for example see [1, 11]), for more applications see [13, 25, 29, 33 and 34].

There are many researchers who used various numerical methods to solve convection-diffusion equation. Appadu [2] solved the convection-diffusion equation using standard and nonstandard finite difference schemes. Chawla et al. [9] presented extended one-step time-integration schemes for convection–diffusion equations. El-Hawary and Abdel-Rahman [16] studied numerical solution of convection-diffusion equation (linear Burger's) by a spectral/spline method. EL-Wakil and Elhanbaly [17] solved the convection-diffusion equation using Adomian decomposition method. Feng [19], explicit finite difference method was used to solve convection-diffusion equation. Olayiwola [27] presented Variational iteration method to solve convection-diffusion equation. In Porshokouhi et al. [30], Homotopy perturbed method was used to solve convection-diffusion equation. Temsah [31] presented steady-state solution for

Received: 31 January 2018, Accepted: 2 June 2018

convection-diffusion equation using El-Gendi method. For more applications to obtain the numerical solutions of convection-diffusion equation see [4, 8, 10, 12, 14, 18, 20, 26 and 32]. In this work, we use the spectral method based on expanding the unknown functions in terms of Chebyshev polynomials of order N. The purpose of this paper is to find a numerical solution of convection-diffusion equation by using Chebyshev expansion method, which is a continuations of our previous work [3], in which we found the exact solution of the given equation by using Iie group method. For the applications of Iie group method refer to [6, 7, 21, 22, 23, 24 and 28].

2. PROBLEM DEFINITION

2.1 METHOD OF CHEBYSHEV EXPANSION

The Chebyshev polynomial of the first kind of degree n, $\mathrm{T}_n(x)$, has $(n+1)$ extreme located in the interval $[-1,1]$ at the points

$$x_i = \cos(i\pi/N),\ i = 0,1,\cdots,N \qquad (4)$$

Assume that the function $f(x)$ is defined and well-behaved in the interval $(-1,1)$. We have the approximation

$$f(x) = \sum_{r=0}^{N}{}'' a_r \mathrm{T}_r(x) \qquad (5)$$

where, the Chebyshev coefficients a_r are defined as

$$a_r = \frac{2}{N}\sum_{i=0}^{N}{}'' f(x_i)\mathrm{T}_r(x_i),\quad r = 0,1,\cdots,N \qquad (6)$$

Here $\mathrm{T}_r(x)$ is the rth Chebyshev polynomial. A double prime indicates that both the first and last terms of the sum are taken with factor $\frac{1}{2}$. The approximate formula (5) is exact at $x = x_i$ given by (4).

If we use (6) in (5), then after certain manipulations we have the following approximation

$$\int_{-1}^{x} f(x)\,dx = B[f], \qquad (7)$$

where B is an $(N+1)\times(N+1)$ matrix (see [15]), the elements of the column $[f]$ are given by $f(x_i)$ where x_i are the Chebyshev points

$$x_i = -\cos(i\pi/N),\ i = 0,1,\cdots,N \qquad (8)$$

The right-hand side of equation (7) gives approximations of the integral at the points (4). In other words, we evaluate the integral at the points (8) rather than evaluate the Chebyshev coefficients of the integral. The two approximations are equivalent in the sense that if we know the integral at the points (8), its Chebyshev coefficients can be directly evaluated from (6). We note that the computation of the elements of B does not depend on the integral. Note: For the interval (0,1) the elements of matrix B are to be halved, and the Chebyshev points will be $x_i = \frac{1}{2}(1-\cos(i\pi/N)),\ i = 0,1,\cdots,N$ such that the Chebyshev polynomials are given by

$$\mathrm{T}^*_r(x) = \mathrm{T}_r(2x-1),\quad 0 \le x \le 1 \qquad (9)$$

where, all the properties of $\mathrm{T}^*_r(x)$ can be deduced from those of $\mathrm{T}_r(2x-1)$.

3. LIE GROUP TRANSFORMATION METHOD

The aim of this method is used to transform the original partial differential equation (PDE) to ordinary differential equation (ODE). [3] presented the method of Lie group of transformations, which makes convection-diffusion equation invariant. The one-parameter ε-Lie group point of transformations are given by

$$\bar{x} = \bar{x}(x,t,u;\varepsilon) = x + \varepsilon\xi_1(x,t,u) + O(\varepsilon^2),$$
$$\bar{t} = \bar{t}(x,t,u;\varepsilon) = t + \varepsilon\xi_2(x,t,u) + O(\varepsilon^2),$$
$$\bar{u} = \bar{u}(x,t,u;\varepsilon) = u + \varepsilon\varphi(x,t,u) + O(\varepsilon^2) \quad (10)$$

The infinitesimal generator for (10) is given by

$$X = \xi_1\frac{\partial}{\partial x} + \xi_2\frac{\partial}{\partial t} + \varphi\frac{\partial}{\partial u} \quad (11)$$

and the second prolongation of the infinitesimal generator (11) is given by

$$Pr^{(2)}X = X + \varphi^x\frac{\partial}{\partial u_x} + \varphi^t\frac{\partial}{\partial u_t} + \varphi^{xt}\frac{\partial}{\partial u_{xt}} + \varphi^{xx}\frac{\partial}{\partial u_{xx}} + \varphi^{tt}\frac{\partial}{\partial u_{tt}} \quad (12)$$

The components $\varphi^x, \varphi^t, \varphi^{xt}, \varphi^{xx}$ and φ^{tt} can be determined from the following expressions:

$$\varphi^\alpha = D_\alpha\varphi - u_x D_\alpha\xi_1 - u_t D_\alpha\xi_2,$$
$$\varphi^{J\alpha} = D_\alpha\varphi^J - u_{Jx} D_\alpha\xi_1 - u_{Jt} D_\alpha\xi_2,$$
$$D_\alpha \equiv \frac{\partial}{\partial\alpha} + u_\alpha\frac{\partial}{\partial u} + u_{\alpha\alpha}\frac{\partial}{\partial u_\alpha} + u_{\alpha\beta}\frac{\partial}{\partial u_\beta} + \ldots \quad (13)$$

where α, J and β stand for independent variables x,t, considered

$$\Delta = \frac{\partial u}{\partial t} + \alpha\frac{\partial u}{\partial x} - \varepsilon\frac{\partial^2 u}{\partial x^2} \quad (14)$$

To calculate the infinitesimals ξ_1, ξ_2 and φ, applying (12) to (14), i. e.,

$$Pr^{(2)}X(\Delta) = 0, \quad when \ \Delta = 0 \quad (15)$$

Equation (15) is called invariance condition, and leads to

$$\varphi^t + \alpha\varphi^x - \varepsilon\varphi^{xx} = 0, \quad (16)$$

Where φ^t, φ^x and φ^{xx} are obtained from (13). The infinitesimals solutions ξ_1, ξ_2 and φ are given by

$$\xi_1(x,t,u) = \frac{1}{2}xtc_1 + \frac{1}{2}(\alpha t + x)c_2 - 2\varepsilon tc_5 + c_8,$$
$$\xi_2(x,t,u) = \frac{1}{2}t^2c_1 + tc_2 + c_3,$$
$$\varphi(x,t,u) = c_6\exp\left(t + \frac{1}{2\varepsilon}(\alpha - \sqrt{\alpha^2+4\varepsilon})x\right) + c_7\exp\left(\alpha t + \frac{1}{2\varepsilon}(\alpha + \sqrt{\alpha^2+4\varepsilon})x\right)$$
$$+ uc_4 - (\alpha t - x)uc_5 - \frac{1}{4}\left(t - \frac{1}{2\varepsilon}(\alpha t - x)^2\right)uc_1 \quad (17)$$

where $c_i, i = 1,2,\ldots,8$ are arbitrary constants.

From equation (17), the convection-diffusion equation has the following generators

$$X_1 = \frac{1}{2}xt\partial_x + \frac{1}{2}t^2\partial_t - \frac{1}{4}\left(t - \frac{1}{2\varepsilon}(\alpha t - x)^2\right)u\partial_u,$$
$$X_2 = \frac{1}{2}(\alpha t + x)\partial_x + t\partial_t,$$
$$X_3 = \partial_t, \ X_4 = u\partial_u, X_5 = -2\varepsilon t\partial_x - (\alpha t - x)u\partial_u,$$
$$X_6 = \exp\left(t + \frac{1}{2\varepsilon}(\alpha - \sqrt{\alpha^2+4\varepsilon})x\right)\partial_u,$$
$$X_7 = \exp\left(t + \frac{1}{2\varepsilon}(\alpha + \sqrt{\alpha^2+4\varepsilon})x\right)\partial_u, \ X_8 = \partial_x \quad (18)$$

As in [3], we use the linear combination of generators X_3 and X_8 as follows

$$X_{3,8} = \partial_t + c\partial_x \ , \quad (19)$$

and getting of the following similarity reductions:

$$\eta = x - ct,$$
$$f(\eta) = u(x,t) \quad (20)$$

Equation (20) reduce the original equation (1) to the ordinary differential equation:

$$\varepsilon f''(\eta) + (c - \alpha)f'(\eta) = 0, \quad \eta \in [-ct, 1-ct] \quad (21)$$

In this work, without loss of the generality, we can choose $c = 1$, and equation (21) becomes

$$\varepsilon f''(\eta) + (1 - \alpha)f'(\eta) = 0, \quad \eta \in [-t, 1-t] \quad (22)$$

with the moving boundary conditions

$$f = g_0(t) + \varepsilon_0 \ at \ \eta \to -t, \quad f = g_1(t) + \varepsilon_1 \ at \ \eta \to 1 - t \quad (23)$$

Using the algebraic mapping, $y = \eta + t$,

the boundary region is mapped into the finite domain (0,1), and the problem expressed by equations (22) with (23) is transformed to

$$\varepsilon f'' + (1 - \alpha)f' = 0 \quad (24)$$

$$f(0) = g_0(t) + \varepsilon_0,$$
$$f(1) = g_1(t) + \varepsilon_1 \quad (25)$$

where the prime denotes differentiation with respect to y .Our technique is accomplished by starting with a Chebyshev approximation for the higher order derivatives f'' and generating approximations to the lower derivatives f' and f as follows. Setting $\psi(y) = f''(y)$, then by integration we obtain

$$f'(y) = \int_0^y \psi(y)dy + c_1, \tag{26}$$

$$f(y) = \int_0^y \int_0^y \psi(y)\,dy\,dy + yc_1 + c_2 \tag{27}$$

From equation (27), we get

$$c_1 = g_1(t) + \varepsilon_1 - (g_0(t) + \varepsilon_0) - \int_0^1 \int_0^y \psi(y)\,dy\,dy,$$

$$c_2 = g_0(t) + \varepsilon_0$$

we give Chebyshev approximations for equations (26) and (27) as follows:

$$f'(y_i) = \sum_{j=0}^{N} \ell_{ij}^{(1)} \psi(y_j) + d_i^{(1)}, \tag{28}$$

$$f(y_i) = \sum_{j=0}^{N} \ell_{ij} \psi(y_j) + d_i \tag{29}$$

for all $i, j = 0,1,\ldots,N$ and $y_i = \frac{1}{2}(1 - \cos(i\pi/N))$ where ,

$$\ell_{ij} = b_{ij}^{(2)} - y_i b_{Nj}^{(2)},\ \ell_{ij}^{(1)} = b_{ij} - b_{Nj}^{(2)},$$

$$d_i = (g_1(t) + \varepsilon_1 - (g_0(t) + \varepsilon_0))y_i + g_0(t) + \varepsilon_0,$$

$$d_i^{(1)} = g_1(t) + \varepsilon_1 - (g_0(t) + \varepsilon_0)$$

and $b_{ij}^{(2)} = (y_i - y_j)b_{ij},\ \ i, j = 0,1,\ldots,N.$

where, b_{ij} are the elements of the matrix $\frac{1}{2}$B as defined by El-Gendi [15] by using equations (28) and (29). Equation (24) is transformed to the following nonlinear system:

$$\psi_i + \frac{1-\alpha}{\varepsilon}\left(\sum_{j=0}^{N} \ell_{ij}^{(1)} \psi(y_j) + d_i^{(1)} \right) = 0 \tag{30}$$

This system is solved by using Gauss elimination method to obtain the approximate solution of the original system.

4. RESULTS AND DISCUSSION

4.1 NUMERICAL EXAMPLES

In this section, we present some examples of convection-diffusion equation and we compare our numerical results with the exact solutions, see Tables 1 to 3. Computations of the examples have been carried out using the FORTRAN. In Table 1, we present absolute errors for various values of x, t with $\varepsilon_0 = 0$, $\varepsilon_1 = 0$ (without perturbations) and N=18 for example 1.The maximum absolute error for various values of t with $\varepsilon = 1,\ \varepsilon_0 = 0$, $\varepsilon_1 = 0$ and N=12 are shown in Table 2 for example 2. Table 3 shows absolute errors in the results obtained by the proposed method for various values of x, t with $\varepsilon_0 = 10E-15$, $\varepsilon_1 = 10E-13$ and N=18 for example 3. Example 1, consider the convection-diffusion equation in (2)

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - 0.01\frac{\partial^2 u}{\partial x^2} = 0$$

where, $f(x) = \exp(-\frac{(x+0.5)^2}{0.00125})$

$$g_0(t) = \frac{0.025}{\sqrt{0.00025 + 0.02t}} \exp\left(-\frac{(0.5-t)^2}{(0.00125 + 0.04t)} \right)$$

$$g_1(t) = \frac{0.025}{\sqrt{0.00025 + 0.02t}} \exp\left(-\frac{(1.5-t)^2}{(0.00125 + 0.04t)} \right)$$

The exact solution is

$$u(x,t) = \frac{0.025}{\sqrt{0.00025 + 0.02t}} \exp\left(-\frac{(x+0.5-t)^2}{(0.00125 + 0.04t)} \right)$$

Example 2**,** consider the convection-diffusion equation in (5)

$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}-\varepsilon\frac{\partial^2 u}{\partial x^2}=0$, and the exact solution is

$$u(x,t)=\frac{1}{\sqrt{s}}\exp\left(-50\frac{(x-t)^2}{s}\right),\quad s=1+200\varepsilon t$$

where, the initial condition $u(x,0)$ and boundary conditions $u(x,t)$ at $x=0,1$ without perturbations are obtained from the exact solution. Example 3, consider the convection-diffusion equation in (30)

$$\frac{\partial u}{\partial t}+0.22\frac{\partial u}{\partial x}-0.5\frac{\partial^2 u}{\partial x^2}=0$$

where, $f(x)=\sin(\pi x)\,e^{0.22x}$

$$g_0(t)=0=g_1(t)$$

The exact solution is

$u(x,t)=\sin(\pi x)\,e^{0.22x-(0.0242+0.5\pi^2)t}$.

TABLE 1: THE ABSOLUTE ERRORS BETWEEN EXACT AND NUMERICAL SOLUTION FOR EXAMPLE 1.

x	t			
	0.01	0.05	0.08	0.1
0.25	0.000000E+00	4.052167E-28	2.419396E-18	2.127136E-14
0.5	0.000000E+00	2.701445E-28	1.612931E-18	1.418091E-14
0.75	0.000000E+00	1.350722E-28	8.064654E-19	7.090453E-15

TABLE 2: THE MAXIMUM ABSOLUTE ERRORS FOR EXAMPLE 2

t	1.0	1.5	2.0	2.5	3.0
	3.514044E-03	1.338173E-03	5.243644E-04	1.703687E-04	3.712066E-05

TABLE 3: THE ABSOLUTE ERRORS BETWEEN EXACT AND NUMERICAL SOLUTION FOR EXAMPLE 3

x	t						
	1	2	3	4	5	6	7
0.25	5.24450E-03	3.68160E-05	2.58445E-07	1.81426E-09	1.27318E-11	8.51697E-14	3.60866E-15
0.5	7.83619E-03	5.50095E-05	3.86163E-07	2.71083E-09	1.90225E-11	1.26259E-13	6.39146E-15
0.75	5.85432E-03	4.10969E-05	2.88497E-07	2.02522E-09	1.42077E-11	9.05707E-14	8.53068E-15

5. CONCLUSION

The numerical results obtained by Chebyshev approximation method based on El-Gendi method via Lie group method for convection- diffusion equation show reasonable agreement with the exact solution as shown in Tables 1-3. Consequently, the presented method performs well.

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