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Research article

A new study of relativistic and nonrelativistic for new modified Yukawa potential via the BSM in the framework of noncommutative quantum mechanics symmetries: An application to heavy-light mesons systems

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In this article, the deformed Dirac equation was solved with a new modified Yukawa potential within three tensor interactions (Coulomb-like, Yukawa-like, and Hulthén-type potentials) (NMYTTIs). This study is scrutinized by taking into consideration the noncommutative space-space effect within the relativistic and nonrelativistic regimes using the parametric of Bopp's shift method and standard perturbation theory with the help of an approximation to the centrifugal term. In the framework of the spin and pseudospin (p-spin) symmetry, we obtain the global new energy eigenvalue which equals the energy eigenvalue in usual relativistic QM under the new modified Yukawa potential plus the corrected of energy induced with three infinitesimal additive parts of the relativistic Hamiltonian corresponds to the spin-orbit interaction, the new modified Zeeman and the rotational Fermi term. The new values that we get appeared sensitive to the quantum numbers $(j,\ k,\ l,\ l^{\mathrm{ps}},\ s,\ s^{\mathrm{ps}},\ m,\ m^{\mathrm{ps}})$ and $(j,\ l,\ s,\ m)$ (within the relativistic and nonrelativistic regimes, respectively), the potential depths $(A,\ B,\ C)$, the range of the potential α, and noncommutativity parameters $(\eta,\ \sigma,\ \chi)$. The present results are applied for calculating the mass spectra of heavy mesons such as heavy-light mesons HLM such as $c\overline{c}$, $b\overline{b}$, $b\overline{c}$, $b\overline{s}$, $c\overline{s}$ and $b\overline{q}$, $q=(u,\ d)$. Our results excellent agree perfectly with the ones in works of literature.

1. INTRODUCTION

Researchers have focused on the study of symmetries prominently over the past decades. Spin symmetry and pseudospin symmetry match have had the most luck in this regard what it takes to understand to explain the. quasi-degeneracy in nuclei between single-nucleon states with the quantum number $(n,\ l,\ j=l+\frac{1}{2})$ and $(n-1,\ l+2,\ j=l+\frac{3}{2})$.[1–3] Furthermore, pseudospin symmetry is used to feature deformed nuclei and superdeformation to establish an effective shell model.[3,4] Many research papers related to the Yukawa potential such as the generalized Yukawa potential,[5] generalized Yukawa problems,[6] the Dirac Mobius square -Yukawa and Mobius square-quasi Yukawa problems,[7] generalized Yukawa potential and tensor terms,[8] inversely quadratic Yukawa-like plus Mobius square potentials including a Coulomb-like tensor interaction[9] relativistic new Yukawa-like potential and tensor coupling,[10] Yukawa field with Coulomb-like interaction,[11] relativistic pseudospin and spin symmetries of the energy-dependent Yukawa potential including a Coulomb-like tensor interaction,[12] the inversely quadratic Yukawa potential and tensor interaction[13] and relativistic symmetries in Yukawa-type interactions with Coulomb-like tensor[14] within the framework of the relativistic Dirac equation focused on the study of these symmetries. Ahmadov et al. (2019)[15] present an analytical solution of the modified radial Schrödinger equation for the sum of Cornell and inverse quadratic potential within quantum mechanics by applying the Nikiforov-Uvarov method and obtaining the energy eigenvalues, mass spectrum, and corresponding eigenfunctions for arbitrary l angular momentum quantum numbers. In 2021, Ahmadov et al.[16] studied the finite temperature-dependent Schrödinger equation by using the Nikiforoto effthev-Uvarov method and considering the sum of the Cornell, inverse quadratic, and harmonic-type poten-

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tials as the potential part of the radial Schrödinger equation and presented analytical expressions for the energy eigenvalues and the radial wave function, and applying the results for the heavy quarkonia and Bc meson masse. Recently, Shakir et al.[17] extended an exactly solvable model of a nonrelativistic quantum linear harmonic oscillator with a position-dependent mass to the case where an external homogeneous gravitational field is applied using a generalized free quantum Hamiltonian with position-dependent mass. Very recently Ahmadov et al.,[18] presented the bound state solutions of the Dirac equation with spin and pseudo-spin symmetries for the Manning–Rosen potential with Yukawa-like tensor interaction using the supersymmetry quantum mechanics and Nikiforov–Uvarov methods framework and obtained obtain the relativistic energy eigenvalues associated with Dirac spinor components of wave functions. It should be noted that there are other studies of great importance that fall into the treatment of spin and pseudospin within the framework of the relative Dirac equation.[19–24] This research aims to shed light with more scrutiny on the study of these symmetries resulting from modified Yukawa potential under relativistic spin and pseudospin symmetries including three-tensor interactions comprising Coulomb-like, Yukawa-like and Hulthén-type potentials from the perspective of a new space includes broader symmetries compared to the space known in the literature. This space is known as a non-commutative space or extended quantum mechanics in the context of Dirac theory we called the deformation Dirac theory DDT and the deformation Schrödinger theory DST for further investigation for new energy values and search for the possibility of discovering new applications. In addition to the well-known axioms that established quantum mechanics known in the literature, we have two additional axioms, the first allocating the non-commutative space- space $x_{\mu}^{\mathrm{nc}(s,\,h,\,i)} * x_{\nu}^{\mathrm{nc}(s,\,h,\,i)} \neq x_{\nu}^{\mathrm{nc}(s,\,h,\,i)} * x_{\mu}^{\mathrm{nc}(s,\,h,\,i)}$ (The symbol $*$ denotes the Weyl-Moyal star product) while the second corresponds to the non-commutative phase-phase $p_{\mu}^{\mathrm{nc}(s,\,h,\,i)} * p_{\nu}^{\mathrm{nc}(s,\,h,\,i)} \neq p_{\nu}^{\mathrm{nc}(s,\,h,\,i)} * p_{\mu}^{\mathrm{nc}(s,\,h,\,i)}$. The researchers believe that the idea of non-commutativity is the best solution to many physical problems that did not find a convincing solution within the framework of quantum mechanics known in the literature such as quantum gravity, string theory, and the divergence problem in the standard model.[25–34] On the other hand, the theory of non-commutativity is a very strong candidate to be the physical tool that unites quantum mechanics with its three interactions (strong-electromagnetic and weak) with gravitational interactions represented in Einstein's general relativity and the nonsymmetric gravitational theory of Moffat. The idea of extended quantum mechanics EQM is not new but goes back decades and was suggested by Snyder[35] in 1947 and its geometric analysis was introduced by Connes in 1991 and 1994.[36,37] Seiberg and Witten, extend earlier ideas about the appearance of NC geometry in string theory with a nonzero B-field and obtained a new version of gauge fields on noncommutative gauge theory.[38] It should be noted that the great mathematical progress during the past decades has encouraged researchers to rely on non-commutative quantum mechanics to delve deeper into understanding atomic and nuclear physical phenomena and to study the interaction between molecules as well. In this context, we have made new contributions in the past years, for example.[39–47] I hope that through this study to discover more investigations in the sub-atomic scales and from achieving more scientific knowledge of elementary particles in the field of Nano-scales. Motivated by previous works in literature, we will focus on a new molecular potential, we called it a new modified Yukawa potential within three tensor interactions (Coulomb-like, Yukawa-like, and Hulthén-type potentials) (*NMYTTIs*) model ($V_{\mathrm{my}}\left(r_{\mathrm{nc}}\right)$, $S_{\mathrm{my}}\left(r_{\mathrm{nc}}\right)$) studied in this work is of the form:

$$\begin{cases} V_{\mathrm{my}}\left(r_{\mathrm{nc}}\right) = V_{\mathrm{my}}(r) - \frac{\partial V_{\mathrm{my}}(r)}{\partial r}\frac{L\eta}{2r} + O\left(\eta^2\right) \\ S_{\mathrm{my}}\left(r_{\mathrm{nc}}\right) = S_{\mathrm{my}}(r) - \frac{\partial S_{\mathrm{my}}(r)}{\partial r}\frac{L\eta}{2r} + O\left(\eta^2\right) \end{cases} \tag{1}$$

where $(V_{\mathrm{my}}(r),\ S_{\mathrm{my}}(r))$ are the vector and scalar potentials[3] according to the view of RQM known in the literature:

$$\begin{cases} V_{\mathrm{my}}(r) = A\frac{\exp(-2\alpha r)}{r^2} - B\frac{\exp(-\alpha r)}{r} + C \\ S_{\mathrm{my}}(r) = A_s\frac{\exp(-2\alpha r)}{r^2} - B_s\frac{\exp(-\alpha r)}{r} + C_s \end{cases} \tag{2}$$

where$A\left(A_s\right)$, $B\left(B_s\right)$ and $C\left(C_s\right)$ are some parameters of modified Yukawa potential, α is the screening parameter, (r_{nc} and r) is the distance between the two particles in deformation of Dirac theory symmetries and QM symmetries, respectively. The coupling $L\eta$ is the scalar product of the usual components of the angular momentum operator $L\left(L_x,\ L_y,\ L_z\right)$ and the modified noncommutativity vector$\eta\frac{(\theta_{12},\,\theta_{23},\,\theta_{13})}{2}$ which present is the noncommutativity elements parameter. The modified algebraic structure of the covariant canonical commutation relations MASCCCRs, the canonical structure (CS), the Lie structure (LS) and the quantum plane (QP) in the *DDT* in the representations of Schrödinger, Heisenberg, and interactions pictures, as follows (we have used the natural units $\hbar = c = 1$)[48–56]:

$$\left[x_{\mu}^{(s,\,h,\,i)},\ p_{\nu}^{(s,\,h,\,i)}\right] = i\hbar\delta_{\mu\nu} \text{ and } \left[x_{\mu}^{(s,\,h,\,i)},\ x_{\nu}^{(s,\,h,\,i)}\right] = 0 \Rightarrow \tag{3}$$

and

$$\begin{gathered} \left[x_{\mu}^{\mathrm{nc}(s,h,i)}, p_{\nu}^{\mathrm{nc}^{(s,h,i)}}\right] = i\hbar_{\mathrm{eff}}\delta_{\mu\nu} \\ \left[x_{\mu}^{\mathrm{nc}^{(s,h,i)}}, x_{\nu}^{\mathrm{nc}(s,h,i)}\right] \\ = \begin{cases} i\theta_{\mu\nu} \text{ with } \theta_{\mu\nu} \in \mathrm{IC},\ \mathrm{CS}, \\ \mathrm{if}_{\mu\nu}^{\alpha}\, x_{\alpha}^{\mathrm{nc}(s,\,h,\,i)} \text{ with } f_{\mu\nu}^{\alpha} \in \mathrm{IC},\ \mathrm{LS}, \\ \mathrm{iC}_{\mu\nu}^{\alpha\beta} x_{\alpha}^{\mathrm{nc}(s,\,h,\,i)} x_{\beta}^{\mathrm{nc}(s,h,i)} \text{with } C_{\mu\nu}^{\alpha\beta} \in \mathrm{IC},\ \mathrm{QP}. \end{cases} \end{gathered} \tag{4}$$

with $x_{\mu}^{\mathrm{nc}(s,\,h,\,i)} = \left(x_{\mu}^{\mathrm{ncs}}, x_{\mu}^{\mathrm{nch}}, x_{\mathrm{nc}\mu}^{\mathrm{nci}}\right)$ and $p_{\mathrm{nc}\mu}^{(s,\,h,\,i)} = \left(p_{\mu}^{\mathrm{ncs}}, p_{\mu}^{\mathrm{nch}}, p_{\mu}^{\mathrm{nci}}\right)$ are the generalized coordinates and the corresponding generalizing coordinates in the *DDT* symmetries while IC denoting the complex number field. While the uncertainty relations will be changed into the following formula in the new symmetries as follows:

$$\begin{gathered} \left|\Delta x_{\mu}^{(s,\,h,\,i)}\Delta p_{\nu}^{(s,\,h,\,i)}\right| \geq \hbar\frac{\delta_{\mu\nu}}{2} \Rightarrow \\ \begin{cases} \left|\Delta x_{\mu}^{\mathrm{nc}(s,\,h,\,i)}\Delta p_{\nu}^{\mathrm{nc}(s,\,h,\,i)}\right| \geq \hbar_{\mathrm{eff}}\frac{\delta_{\mu\nu}}{2} \\ \left|\Delta x_{\mu}^{\mathrm{nc}(s,\,h,\,i)}\Delta x_{\nu}^{\mathrm{nc}(s,\,h,\,i)}\right| \geq \frac{|\theta_{\mu\nu}|}{2} \end{cases} \end{gathered} \tag{5}$$

It is important to note that Eqs. (3) and (4) are covariant equations (the same behavior $x_{\mu}^{\mathrm{nc}(s,\,h,\,i)}$) under Lorentz transformation, which includes boosts and/or rotations of the observer's inertial frame. We extended the modified

equal-time noncommutative canonical commutation relations to include the Heisenberg and interaction pictures in *DDT*. Here $\hbar_{\rm eff} \simeq \hbar$ is the effective Planck constant, $\theta_{\mu\nu} = \varepsilon_{\mu\nu}\theta$ (θ is the NC parameter and $\varepsilon_{\mu\nu}$ is just an antisymmetric number $\left(\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu} = 1 \text{ with } \mu \neq \nu\right)$ and $\varepsilon_{\varepsilon\varepsilon} = 0$) which is an infinitesimal parameter if compared to the energy values and elements of antisymmetric (3×3) real matrices and $\delta_{\mu\nu}$ is the Kronecker symbol. The new deformed product form $h(x^{\rm nc})g(x^{\rm nc})$ can be expressed with the Weyl-Moyal star product $h(x) * g(x)$ in the symmetries of *DDT* symmetries as follows[57–61]:

$$h(x) * g(x) = \begin{cases} \exp\left(i\varepsilon^{\mu\nu}\theta\partial_\mu^x\partial_\nu^x\right)\ (\text{hg})\ (x)\ , \text{CS}, \\ \exp\left(\frac{i}{2}x_{\text{nc}\mu}^{(s,\,h,\,i)}g_k\left(i\partial_\mu^x,\ i\partial_\nu^x\right)\right)\ (\text{hg})\ (x)\ , \text{LS}, \\ \text{iq}^{G(u,\,v,\partial_\mu^u,\partial_\nu^v)}h(u,\ v)\ g\left(u^{'},\ v^{'}\right)|_{u^{'}\to u}^{v^{'}\to v}, \text{QP}. \end{cases} \quad (6)$$

with

$$g_\alpha(k,\ p) = -k_\mu p_\nu f_k^{\nu\nu} + \frac{1}{6}k_\mu p_\nu\left(p_\alpha - k_\alpha\right) f_l^{\nu\nu} f_m^{l\alpha} + ...$$

In the current paper, we apply the modified algebraic structure of covariant canonical commutation relations MASCCCRs in the DDT, which allows us to rewrite to the following simple form at the first order of noncommutativity parameter$\varepsilon^{\mu\nu}\theta$ as follows[60–63]:

$$(h * g)\ (x) = (\text{hg})\ (x) - \frac{i\varepsilon^{\mu\nu}\theta}{2}\partial_\mu^x h \partial_\nu^x g|_{x^\mu = x^\nu} + O\left(\theta^2\right) \quad (7)$$

The indices$(\mu,\ \nu = 1,\ 2,\ 3)$ and $O\left(\theta^2\right)$ stand for the second and higher-order terms of the NC parameter. Physically, the second term in the last equation presents the effects of space-space non-commutativity. The main aim of the paper is to investigate the $(k,\ l)$-states solutions of the deformed Dirac equation with the *NMYTTIs* model in the symmetries of *DDT*, within the frame of the parametric Bopp shift method (BSM). The present paper is organized as follows. The first section includes the scope and purpose of our investigation, while the remaining parts of the paper are structured as follows. A review of the DE with the modified Yukawa potential with three improved tensor interactions is presented in Sect. 2. Sect. 3 is devoted to studying the deformed Dirac equation by applying the usual BSM and the Greene-Aldrich approximation for the centrifugal terms to obtain the effective potentials of the *NMYTTIs* model in DDT symmetries. Furthermore, via standard perturbation theory, we find the expectation values of some radial terms to calculate the corrected relativistic energy generated by the effect of the perturbed effective potentials of the *NMYTTIs* model, we derive the global corrected energy with the *NMYTTIs* model in the DDT symmetries. We will also treat some important special cases, including the study of relativistic cases as a non-relativistic limit in the next section. The present results are applied for calculating the mass spectra of heavy mesons such as $c\overline{c}$,$b\overline{b}$, $b\overline{c}$, $b\overline{s}$, $c\overline{s}$ and $b\overline{q}$, q= $(u,\ d)$. Section six is devoted to the conclusions.

2. AN OVERVIEW OF DE UNDER *MYP*

This section is devoted to a brief review of a physical system that interacted with modified Yukawa potential MYP within three tensor interactions comprising Coulomb-like, Yukawa-like, and Hulthén-type potentials, this system can be described by the following equation:

$$\mathbf{H}_D^{\rm my}\Psi_{\rm nk}(r,\ \theta,\ \phi) = E_{\rm nk}\Psi_{\rm nk}(r,\ \theta,\ \phi) \quad (8)$$

here

$$\mathbf{H}_D^{my} = c\boldsymbol{\alpha}\mathbf{p} + \boldsymbol{\beta}\left(Mc^2 + S_{my}(r)\right) - i\beta\mathbf{r}U(r) + V_{my}(r)$$

is the Dirac Hamiltonian operator, M is reduced rest mass, $\mathbf{p} = -\mathbf{i}\hbar\nabla$ is the momentum. The vector potential $V_{\rm my}(r)$ due to the four-vector linear momentum operator $A^\mu(V_{\rm my}(r),\ \mathbf{A} = 0)$ and space-time scalar potential $S_{\rm my}(r)$ due to the mass, $E_{\rm nk}$ represents the relativistic eigenvalues, $(n,\ k)$ representing the principal and spin-orbit coupling terms, respectively. The tensor interaction $U(r)$ is composed with mixed of three potentials given by:

$$U(r) = -\frac{1}{r}(H_c + H_Y\exp(-\infty\cdot)) - H_H\frac{\exp(-\alpha r)}{1-\exp(-\alpha r)}$$

here H_c, H_Y and H_H are the Coulomb, Yukawa, and Hulthén parameters, the effect of mixed-like tensor terms has been reported to remove the degeneracies in the Dirac theory, $\alpha_i = \text{anti} - \text{diag}(\sigma_i,\ \sigma_i)$, $\beta = \text{diag}(I_{2\times2}, -I_{2\times2})$ and σ_i are the usual Dirac matrices. Since the MYP has spherical symmetry, the solutions of the known form are allowed:

$$\Psi_{nk}(r,\theta,\varphi) = \begin{pmatrix} \frac{F_{nk}(r)}{r}Y_{jm}^{l}(\theta,\varphi) \\ i\frac{G_{nk}(r)}{r}Y_{jm}^{l^{ps}}(\theta,\varphi) \end{pmatrix}$$

The two functions $F_{\rm nk}(r)$and $G_{\rm nk}(r)$represent the upper and lower components of the Dirac spinors $\Psi_{\rm nk}(r,\ \theta,\ \phi)$ while $Y_{\rm jm}^{l}(\theta,\ \phi)$ and $Y_{\rm jm}^{l^{\rm ps}}(\theta,\ \phi)$ are the spin and pseudospin spherical harmonics and m is the projection on the z-axis. The upper and lower components $F_{\rm nk}(r)$ and $G_{\rm nk}(r)$ satisfying the two uncoupled differential equations as below:

$$\begin{bmatrix} \frac{d^2}{dr^2} - k(k+1)r^{-2} + U_{eff}^{glt-s}(r) - (M + E_{nk} - \Delta_{my}(r)) \\ (M - E_{nk} + \Sigma_{my}(r)) + \frac{\frac{d\Delta_{my}(r)}{dr}\left(\frac{d}{dr} + \frac{k}{r} - U(r)\right)}{M + E_{nk} - \Delta_{my}(r)} \end{bmatrix} F_{nk}(r) = 0 \quad (9.1)$$

and

$$\begin{bmatrix} \frac{d^2}{dr^2} - k(k-1)r^{-2} + U_{eff}^{glt-p}(r) - (M + E_{nk} - \Delta_{my}(r)) \\ (M - E_{nk} + \Sigma_{my}(r)) + \frac{\frac{d\Sigma_{my}(r)}{dr}\left(\frac{d}{dr} + \frac{k}{r} + U(r)\right)}{M + E_{nk} + \Sigma_{my}(r)} \end{bmatrix} G_{nk}(r) = 0 \quad (9.2)$$

here

$$U_{eff}^{glt-s/p}(r) = \frac{2kU(r)}{r} \mp \frac{dU(r)}{dr} - U^2(r),$$

After a straightforward calculation, we can express $U_{\rm eff}^{\rm glt-\frac{s}{p}}(r)$ as

$$\begin{aligned} U_{\rm eff}^{\rm glt-\frac{s}{p}}(r) = \frac{a_1^{\mp}}{r^2} + a_2\frac{\exp(-\alpha r)}{r^2} \\ + b_1^{\mp}\frac{\exp(-\alpha r)}{r} + b_2\frac{\exp(-2\alpha r)}{r^2} \\ + a_3\frac{\exp(-\alpha r)}{r(1-\exp(-\alpha r))} + a_4^{\mp}\frac{\exp(-2\alpha r)}{(1-\exp(-\alpha r))^2} \\ + a_5^{\mp}\frac{\exp(-\alpha r)}{1-\exp(-\alpha r)} + a_6\frac{\exp(-2\alpha r)}{r(1-\exp(-\alpha r))} \end{aligned} \quad (10)$$

with $a_1^{\mp} = -\left(2\text{kH}_c + H_c^2 \pm H_c^2\right)$, $a_2 = -2H_cH_Y$, $b_1^{\mp} = \mp\alpha H_Y \mp H_Y$, $b_2 = -\left(2\text{kH}_Y - H_Y^2\right)$, $a_3 = -\left(2\text{kH}_H + 2H_HH_c\right)$, $a_4^{\mp} = -\left(H_H^2 \pm \alpha H_H\right)$, $a_5 = \mp\alpha H_H$ and $a_6 = -2H_HH_Y$ while $\Sigma_{\rm my}(r) = V_{\rm my}(r) + S_{\rm my}(r)$ and $\Delta_{\rm my}(r) = V_{\rm my}(r) - S_{\rm my}(r)$ are determined by:

$$\begin{cases} \Sigma_{\rm my}(r) = A\frac{\exp(-2\alpha r)}{r^2} - B\frac{\exp(-\alpha r)}{r} \text{and } \frac{d\Delta_{\rm my}(r)}{dr} = 0 \\ (\Delta = C_s)\ \text{Spin sy. limit} \\ \Delta_{\rm my}(r) = A\frac{\exp(-2\alpha r)}{r^2} - B\frac{\exp(-\alpha r)}{r} \text{ and } \frac{d\Sigma_{\rm my}(r)}{dr} = 0 \\ (\Sigma = C_{\rm ps})\ \text{P-spin sy. limit} \end{cases} \quad (11)$$

We obtain the following second-order Schrödinger-like equation in RQM symmetries, respectively:

$$\begin{bmatrix} \frac{d^2}{dr^2} - k(k+1)r^{-2} + U_{eff}^{glt-s}(r) \\ -\beta_s \left(M - E_{nk}^{s} + \Sigma_{my}(r)\right) \end{bmatrix} F_{nk}(r) = 0 \qquad (12)$$

and

$$\begin{bmatrix} \frac{d^2}{dr^2} - k(k-1)r^{-2} + U_{eff}^{glt-p}(r) \\ -\left(M + E_{nk}^{ps} - \Delta_{my}(r)\right)\beta_{ps} \end{bmatrix} G_{nk}(r) = 0 \qquad (13)$$

with $k(k-1)$ and $k(k+1)$ are equals $l^{\rm ps}\,(l^{\rm ps}-1)$ and $l(l+1)$, respectively. The authors of Ref.[3] used the NUM method and Greene-Aldrich approximation for the centrifugal term to obtain the expressions for the wave function as hypergeometric polynomials $P_n^{\left(2\sqrt{W^s},\,2\chi_{\rm nk}^s\right)}(1-2s)$ and $P_n^{\left(2\sqrt{W^{\rm ps}},\,2\chi_{\rm nk}^{\rm ps}\right)}(1-2s)$ in RQM symmetries as,

$$F_{\rm nk}(s) = N_{\rm nk}^s s^{\sqrt{W^s}}(1-s)^{\frac{1}{2}+\chi_{\rm nk}^s} P_n^{\left(2\sqrt{W^s},\,2\chi_{\rm nk}^s\right)}(1-2s) \qquad (14)$$

$$G_{\rm nk}(s) = N_{\rm nk}^{\rm ps} s^{\sqrt{W^{\rm ps}}}(1-s)^{\frac{1}{2}+\chi_{\rm nk}^{\rm ps}} P_n^{\left(2\sqrt{W^{\rm ps}},\,2\chi_{\rm nk}^{\rm ps}\right)}(1-2s) \qquad (15)$$

with

$\chi_{\rm nk}^s = \sqrt{\frac{1}{4} + L^s + W^s - K^s}$,

$\chi_{\rm nk}^{\rm ps} = \sqrt{\frac{1}{4} + L^{\rm ps} + W^{\rm ps} - K^{\rm ps}}$,

and $s = \exp(-\alpha r)$ while L^s,K^s, W^s,$L^{\rm ps}$,$K^{\rm ps}$, $W^{\rm ps}$ are given by:

$$\begin{cases} L^s = \frac{\varepsilon_s^2}{\alpha^2} + \frac{\beta_s C}{\alpha^2} + \frac{\beta_s B}{\alpha} + \beta_s A + H_Y\left(H_Y - 1\right) \\ \qquad + \left(\frac{H_H}{\alpha} + 2H_Y\right)\frac{H_H}{\alpha}, \\ K^s = \frac{2\varepsilon_s^2}{\alpha^2} + \frac{2\beta_s C}{\alpha^2} + \frac{\beta_s B}{\alpha} - H_Y\left(2k + 2H_c + 2\right) \\ \qquad - \left(2k + 2H_c + 1\right)\frac{H_H}{\alpha}, \\ W^s = \frac{\varepsilon_s^2}{\alpha^2} + \frac{\beta_s C}{\alpha^2} + \left(k + H_c\right)\,\left(k + H_c + 1\right) \\ W^{\rm ps} = \frac{\varepsilon_{\rm ps}^2}{\alpha^2} - \frac{\beta_{\rm ps} C}{\alpha^2} + \left(k + H_c\right)\,\left(k + H_c - 1\right) \\ L^{\rm ps} = \frac{\varepsilon_{\rm ps}^2}{\alpha^2} - \frac{\beta_{\rm ps} C}{\alpha^2} - \frac{\beta_{\rm ps} B}{\alpha} - \beta_{\rm ps} A + H_Y\left(H_Y + 1\right) \\ \qquad + \left(\frac{H_H}{\alpha} + 2H_Y\right)\frac{H_H}{\alpha}, \\ K^{\rm ps} = \frac{2\varepsilon_{\rm ps}^2}{\alpha^2} - \frac{2\beta_{\rm ps} C}{\alpha^2} - \frac{\beta_{\rm ps} B}{\alpha} - H_Y\left(2k + 2H_c - 2\right) \\ \qquad - \left(2k + 2H_c - 1\right)\frac{H_H}{\alpha}. \end{cases} \qquad (16)$$

here

$\varepsilon_s^2 = M^2 - E_{\rm nk}^{s2} + C_s\left(E_{\rm nk}^s - M\right)$,$\varepsilon_{\rm ps}^2 = M^2 - E_{\rm nk}^{\rm ps2} + C_{\rm ps}\left(E_{\rm nk}^{\rm ps} + M\right)$,$\beta_s = M + E_{\rm nk}^s - C_s$, $\beta_{\rm ps} = M - E_{\rm nk}^{\rm ps} + C_{\rm ps}$ while$N_{\rm nk}^s$ and $N_{\rm nk}^{\rm ps}$ are the normalization constants. For the spin symmetry and the p-spin symmetry, the equations of energy are given:

$$\begin{aligned} n^2 + \left(n + \tfrac{1}{2}\right) + (2n+1)\,\left(\chi_{\rm nk}^s + \sqrt{W^s}\right) - K^s \\ + 2W^s + 2\sqrt{W^s}\chi_{\rm nk}^s = 0 \end{aligned} \qquad (17)$$

$$\begin{aligned} n^2 + \left(n + \tfrac{1}{2}\right) + (2n+1)\,\left(\chi_{\rm nk}^s + \sqrt{W^{\rm ps}}\right) - K^{\rm ps} \\ + 2W^{\rm ps} + 2\sqrt{W^{\rm ps}}\chi_{\rm nk}^s = 0 \end{aligned} \qquad (18)$$

From the definition of Jacobi polynomials:

$$\begin{aligned} P_n^{(a_n,\,b_n)}(1-2s) = \frac{\Gamma(n+a_n+1)}{n!\Gamma(a_n+1)} \\ {}_2F_1\left(-n,\;n+a_n+b_n+1;\;1+a_n,\;s\right) \end{aligned} \qquad (19)$$

we obtain $F_{\rm nk}(r)$ and $G_{\rm nk}(r)$ represent the upper and lower components of the Dirac spinors $\Psi_{\rm nk}(r,\,\theta,\,\phi)$ as a function of the wave function as hypergeometric polynomials ${}_2F_1\left(-n,\,Q_{\rm nk}^s;\,V^s,\,s\right)$ and ${}_2F_1\left(-n,\,Q_{\rm nk}^{\rm ps};\,V^{\rm ps},\,s\right)$ as follows:

$$F_{\rm nk}(s) = N_{\rm nk}^{\rm ns} s^{\sqrt{W^s}}(1-s)^{\frac{1}{2}+\chi_{\rm nk}^s}\,{}_2F_1\left(-n,\,Q_{\rm nk}^s;\,V^s,\,s\right) \qquad (20)$$

$$G_{\rm nk}(s) = N_{\rm nk}^{\rm nps} s^{\sqrt{W^{\rm ps}}}(1-s)^{\frac{1}{2}+\chi_{\rm nk}^{\rm ps}}\,{}_2F_1\left(-n,\,Q_{\rm nk}^{\rm ps};\,V^{\rm ps},\,s\right) \qquad (21)$$

with

$Q_{\rm nk}^{(s,\,\rm ps)} = 2\sqrt{W^{(s,\,\rm ps)}} + 2\chi_{\rm nk}^{(s,\,\rm ps)} + n + 1$,$V^{(s,\,\rm ps)} = 1 + 2\sqrt{W^{(s,\,\rm ps)}}$

while $N_{\rm nk}^{\rm ns} = N_{\rm nk}^s \frac{\Gamma\left(n+2\sqrt{W^s}+1\right)}{n!\Gamma\left(2\sqrt{W^s}+1\right)}$ and $N_{\rm nk}^{\rm nps} = N_{\rm nk}^{\rm ps}\frac{\Gamma\left(n+2\sqrt{W^{\rm ps}}+1\right)}{n!\Gamma\left(2\sqrt{W^{\rm ps}}+1\right)}$.

3. THE NEW SOLUTIONS OF DDE UNDER NMYTTIS IN DDT SYMMETRIES

3.1. REVIEW OF BSM

In this subsection, we are going to solve the deformed Dirac equation DDE with *NMYTTIs* by using Bopp's shift method. We obtained *DDE* by applying the new principles which we have seen in the introduction, Eqs. (4) and (7), summarized in new relationships, MASCCCRs, and the notion of the Weyl-Moyal star product. These data allow us to rewrite the usual radial Dirac equations in Eq. (8) in the DDT symmetries as follows:

$$\begin{pmatrix} \alpha\mathbf{p} + \boldsymbol{\beta}\left(M + S_{my}(r)\right) \\ -i\beta\mathbf{r}U(r) - \left(E_{nk} - V_{my}(r)\right) \end{pmatrix} * \Psi_{nk}(r,\theta,\varphi) = 0 \qquad (22)$$

Thus, the upper and lower components $F_{\rm nk}(r)$ and $G_{\rm nk}(r)$ satisfying the following second-order differential equations in the *DDT* symmetries:

$$\begin{bmatrix} \frac{d^2}{dr^2} - k(k+1)r^{-2} + U_{eff}^{glt-s}(r) \\ -\beta_s \left(M - E_{nk}^{s} + \Sigma_{my}(r)\right) \end{bmatrix} * F_{nk}(r) = 0 \qquad (23)$$

and

$$\begin{bmatrix} \frac{d^2}{dr^2} - k(k-1)r^{-2} + U_{eff}^{glt-p}(r) \\ -\left(M + E_{nk}^{ps} - \Delta_{my}(r)\right)\beta_{ps} \end{bmatrix} * G_{nk}(r) = 0 \qquad (24)$$

Among the possible paths to finding solutions to Eqs. (23) and (24) are according to the application of the Connes method,[36,37] or the Seiberg and Witten map.[38] It is known to specialists that the star product can be translated into the ordinary product known in the literature using what is called Bopp's shift method. Bopp, F. was the first to consider pseudo-differential operators obtained from a symbol by the quantization rules $(x,p) \to \left(x^{\rm nc} = x - \frac{i}{2}\partial_p,\, p^{\rm nc} = p + \frac{i}{2}\partial_x\right)$ instead of the ordinary correspondence $(x,p) \to (x^{\rm nc} = x,\; p^{\rm nc} = p + \frac{i}{2}\partial_x)$, respectively. This procedure is known by Bopp's shifts for researchers, and this quantization procedure is known by Bopp quantization.[58,64–69] This method has achieved considerable success in recent years. In the search for solutions to the NR deformed Schrödinger equation DSE is under the influence of several different potentials (See the references[70–76]). The success of this method was not limited to the DSE, but extended to the study of various relativistic physics problems, for example, the deformed KGE (See the references[77–83]), for the DDE (See the references[84–87]) and for the deformed Duffin-Kemmer-Petiau equation DDKPE.[59,78,79] Thus, Bopp's shift method BSM is based on reducing second-order linear differential equations of the DSE, DKG, DDE, and DDKPE with Weyl-Moyal star product to second-order linear differential equations of SE, KGE, DE, and DKPE without Weyl-Moyal star product with simultaneous translation in the space-space. It is worth motioning that BSM permutes to reduce the above equations to the simplest form:

$$\begin{bmatrix} \frac{d^2}{dr^2} - k(k+1)r_{nc}^{-2} + U_{eff}^{glt}(r_{nc}) \\ -\beta_s\left(M - E_{nk}^{s} + \Sigma_{my}(r_{nc})\right) \end{bmatrix} F_{nk}(r) = 0 \qquad (25)$$

and

$$\begin{bmatrix} \frac{d^2}{dr^2} - k(k-1)r_{nc}^{-2} + U_{eff}^{glt}(r_{nc}) \\ -\left(M + E_{nk}^{ps} - \Delta_{my}(r_{nc})\right)\beta_{ps} \end{bmatrix} G_{nk}(r) = 0 \qquad (26)$$

The modified equal-time noncommutative canonical commutation relations with the notion of Weyl-Moyal star product in Eqs. (4) become new form with ordinary known products in literature is as follows (see, e.g.,[58,64–68]):

$$\left[x_{\mu}^{\mathrm{nc}(s,h,i)},\, p_{\nu}^{\mathrm{nc}(s,h,i)}\right] = i\hbar_{\mathrm{eff}}\delta_{\mu\nu} \text{ and } \left[x_{\mu}^{\mathrm{nc}(s,h,i)},\, x_{\nu}^{\mathrm{nc}(s,h,i)}\right] = i\theta_{\mu\nu} \qquad (27)$$

The asymmetrical Bopp shift allows us to express the generalized Hermitian operators ($x_{\mu}^{\mathrm{nc}(s,\,h,\,i)}$ and $p_{\mu}^{\mathrm{nc}(s,\,h,\,i)}$) in the symmetries of deformation Dirac theory on the corresponding parameters ($x_{\mu}^{(s,\,h,\,i)}$ and $p_{\mu}^{(s,\,h,\,i)}$) in ordinary QM as[58,64–68]:

$$\begin{cases} x_{\mu}^{nc(s,h,i)} = x_{\mu}^{(s,h,i)} - \sum_{v=1}^{3} \frac{i\theta_{\mu v}}{2} p_{v}^{(s,h,i)} \\ p_{\mu}^{nc(s,h,i)} = p_{\mu}^{(s,h,i)} \end{cases} \qquad (28)$$

here $x_{\mu}^{(s,\,h,\,i)} = \left(x_{\mu}^{s},\, x_{\mu}^{h},\, x_{\mu}^{i}\right)$ and $p_{\mu}^{(s,\,h,\,i)} = \left(p_{\mu}^{s},\, p_{\mu}^{h},\, p_{\mu}^{i}\right)$ are corresponding coordinates in the RQM symmetries. This allows us to find the operator r_{nc}^{2} equal $r^2 - L\eta$[84–87] while the new operators $V_{\mathrm{my}}(r_{\mathrm{nc}})$, $U_{\mathrm{eff}}^{\mathrm{glt}}(r_{\mathrm{nc}})$, $k(k+1)\, r_{\mathrm{nc}}^{-2}$ and $k(k-1)\, r_{\mathrm{nc}}^{-2}$ in the DDT symmetries, are expressed as:

$$\begin{cases} V_{\mathrm{my}}(r_{\mathrm{nc}}) = V_{\mathrm{my}}(r) - \frac{\partial V_{\mathrm{my}}(r)}{\partial r}\frac{L\eta}{2r} + O\left(\eta^2\right) \\ U_{\mathrm{eff}}^{\mathrm{glt}-s}(r_{\mathrm{nc}}) = U_{\mathrm{eff}}^{\mathrm{glt}-s}(r) - \frac{\partial U_{\mathrm{eff}}^{\mathrm{glt}-s}(r)}{\partial r}\frac{L\eta}{2r} + O\left(\eta^2\right) \\ U_{\mathrm{eff}}^{\mathrm{glt}-p}(r_{\mathrm{nc}}) = U_{\mathrm{eff}}^{\mathrm{glt}-p}(r) - \frac{\partial U_{\mathrm{eff}}^{\mathrm{glt}-p}(r)}{\partial r}\frac{L^p\eta}{2r} + O\left(\eta^2\right) \\ k(k+1)\, r_{\mathrm{nc}}^{-2} = k(k+1)\, r^{-2} + \\ k(k+1)\, r^{-4}\mathbf{L}\boldsymbol{\eta} + O\left(\eta^2\right) \\ k(k-1)\, r_{\mathrm{nc}}^{-2} = k(k-1)\, r^{-2} + \\ k(k-1)\, r^{-4}\mathbf{L}^p\boldsymbol{\eta} + O\left(\eta^2\right) \end{cases} \qquad (29)$$

Substituting Eqs. (25) into Eqs. (21) and (22), we obtain the following two similar Schrödinger equations:

$$\begin{bmatrix} \frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} + U_{eff}^{glt}(r) \\ -\beta_s\left(M - E_{nk}^{s} + \Sigma_{my}(r)\right) - \Sigma_{my}^{pert}(r) \end{bmatrix} F_{nk}(r) = 0 \qquad (30)$$

and

$$\begin{bmatrix} \frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} + U_{eff}^{glt}(r) \\ -\left(M + E_{nk}^{ps} - \Delta_{my}(r)\right)\beta_{ps} - \Delta_{my}^{pert}(r) \end{bmatrix} G_{nk}(r) = 0 \qquad (31)$$

with

$$\Sigma_{my}^{pert}(r) = \left[-\frac{1}{2r}\frac{\partial U_{eff}^{glt}(r)}{\partial r} + \frac{k(k+1)}{r^4} - \frac{\beta_s}{2r}\frac{\partial V_{my}(r)}{\partial r}\right]\mathbf{L}\eta \qquad (32)$$

and

$$\Delta_{my}^{pert}(r) = \left[-\frac{1}{2r}\frac{\partial U_{eff}^{\mathrm{glt}}(r)}{\partial r} + \frac{k(k-1)}{r^4} - \frac{\beta_{ps}}{2r}\frac{\partial V_{my}(r)}{\partial r}\right]\mathbf{L}\eta \qquad (33)$$

By comparing (Eqs. (12) and (13)) and (Eqs. (30) and (31)), we observe two additive potentials $\Sigma_{\mathrm{my}}^{\mathrm{pert}}(r)$ and $\Delta_{\mathrm{my}}^{\mathrm{pert}}(r)$. Moreover, these terms are proportional to the infinitesimal noncommutativity parameter η . From a physical point of view, this means that these two spontaneously generated terms $\Sigma_{\mathrm{my}}^{\mathrm{pert}}(r)$ and $\Delta_{\mathrm{my}}^{\mathrm{pert}}(r)$ as a result of the topological properties of the deformation space-space can be considered very small compared to the fundamental terms $\Sigma_{\mathrm{my}}(r)$ and $\Delta_{\mathrm{my}}(r)$, respectively. A direct calculation gives $\frac{\partial V_{\mathrm{my}}(r)}{\partial r}$ and $\frac{\partial U_{\mathrm{eff}}^{\mathrm{glt}-\frac{s}{p}}(r)}{\partial r}$ as follows:

$$\begin{aligned} \frac{\partial V_{\mathrm{my}}(r)}{\partial r} = & -2\alpha A\frac{\exp(-2\alpha r)}{r^2} - 2A\frac{\exp(-2\alpha r)}{r^3} \\ & + \alpha B\frac{\exp(-\alpha r)}{r} + B\frac{\exp(-\alpha r)}{r^2} \end{aligned} \qquad (34)$$

and

$$\begin{aligned} \frac{\partial U_{\mathrm{eff}}^{\mathrm{glt}-\frac{s}{p}}(r)}{\partial r} = & \frac{-2a_1^{\mp}}{r^3} + a_7^{\mp}\frac{\exp(-\alpha r)}{r^2} - 2a_2\frac{\exp(-\alpha r)}{r^3} \\ & - \alpha b_1^{\mp}\frac{\exp(-\alpha r)}{r} - 2\alpha b_2\frac{\exp(-2\alpha r)}{r^2} - 2b_2\frac{\exp(-2\alpha r)}{r^3} \\ & - \alpha a_3\frac{\exp(-\alpha r)}{r^2(1-\exp(-\alpha r))} + a_3\frac{-\alpha\exp(-\alpha r)}{r(1-\exp(-\alpha r))} \\ & - \alpha a_3\frac{\exp(-2\alpha r)}{r(1-\exp(-\alpha r))^2} - 2\alpha a_4^{\mp}\frac{\exp(-2\alpha r)}{(1-\exp(-\alpha r))^2} \\ & - 2\alpha a_4^{\mp}\frac{\exp(-3r)}{(1-\exp(-\alpha r))^3} - \alpha a_5^{\mp}\frac{\exp(-\alpha r)}{1-\exp(-\alpha r)} \\ & - \alpha a_5^{\mp}\frac{\exp(-2\alpha r)}{(1-\exp(-\alpha r))^2} - a_6\frac{\exp(-2\alpha r)}{r^2(1-\exp(-\alpha r))} \\ & - 2\alpha a_6\frac{\exp(-2\alpha r)}{r(1-\exp(-\alpha r))} - \alpha a_6\frac{\exp(-3\alpha r)}{r(1-\exp(-\alpha r))^2} \end{aligned} \qquad (35)$$

with $a_7^{\mp} = -\left(\alpha a_2 + b_1^{\mp}\right)$. Substituting Eq. (34) and (35) into Eqs. (32) and (33), we obtain spontaneously generated terms $\Sigma_{my}^{pert}(r)$ as follows:

$$\begin{aligned} \Sigma_{my}^{pert}(r) = & \left(\frac{k(k+1)+a_1^{\mp}}{r^4} - \frac{a_7+B}{2}\frac{\exp(-\alpha r)}{r^3}\right. \\ & + a_2\frac{\exp(-\alpha r)}{r^4} + \frac{\alpha b_1^{\mp}-\alpha B}{2}\frac{\exp(-\alpha r)}{r^2} \\ & + \left(\alpha b_2 + \alpha A\right)\frac{\exp(-2\alpha r)}{r^3} + \left(b_2 + A\right)\frac{\exp(-2\alpha r)}{r^4} \\ & + \frac{\alpha a_3}{2}\frac{\exp(-\alpha r)}{r^3(1-\exp(-\alpha r))} - \frac{\alpha a_3}{2}\frac{\exp(-\alpha r)}{r^2(1-\exp(-\alpha r))} \\ & + \frac{\alpha a_3}{2}\frac{\exp(-2\alpha r)}{r^2(1-\exp(-\alpha r))^2} + \alpha a_4^{\mp}\frac{\exp(-2\alpha r)}{r(1-\exp(-\alpha r))^2} \\ & + \alpha a_4^{\mp}\frac{\exp(-3r)}{r(1-\exp(-\alpha r))^3} + \frac{\alpha a_5^{\mp}}{2}\frac{\exp(-\alpha r)}{r(1-\exp(-\alpha r))} \\ & + \frac{\alpha a_5^{\mp}}{2}\frac{\exp(-2\alpha r)}{r(1-\exp(-\alpha r))^2} + \frac{a_6}{2}\frac{\exp(-2\alpha r)}{r^3(1-\exp(-\alpha r))} \\ & \left. + \alpha a_6\frac{\exp(-2\alpha r)}{r^2(1-\exp(-\alpha r))} + \frac{\alpha a_6}{2}\frac{\exp(-3\alpha r)}{r^2(1-\exp(-\alpha r))^2}\right)\mathbf{L}\eta \end{aligned} \qquad (36)$$

Whereas, the generated potential $\Delta_{\mathrm{my}}^{\mathrm{pert}}(r)$ can be obtained by applying the two simultaneous transformations from Eq. (36):

$$\Delta_{my}^{pert}(r) = \sum_{my}^{pert}(r)\left[\begin{array}{l}\beta_s \to \beta_{ps}, \left(a_1^{-}, b_1^{-}, a_4^{-}, a_5^{-}, a_7^{-}\right) \to \\ \left(a_1^{+}, b_1^{+}, a_4^{+}, a_5^{+}, a_7^{+}\right) \text{and } (k+1) \to k(k-1)\end{array}\right] \qquad (37)$$

For spin symmetry, we first consider Eq. (30), which contains the improved midfield Yukawa potential with three improved tensor interactions in the deformation of Dirac theory symmetries. It can be solved exactly only for $k = 0$ and $k = -1$ in the absence of tensor interactions ($H_c = H_Y = H_H = 0$), since the two centrifugal terms (proportional to $k(k+1)\, r^{-2}$ and $k(k+1)\, r^{-4}$) vanish. In the case of arbitrary k, an appropriate approximation should be used in centrifugal terms. We apply the following improved approximation which was applied by Greene and Aldrich[88]:

$$\begin{aligned} \frac{1}{r^2} &\approx \frac{\alpha^2}{(1-e^{-\alpha r})^2} = \frac{\alpha^2}{(1-s)^2} \\ \Leftrightarrow \frac{1}{r} &\approx \frac{\alpha}{1-e^{-\alpha r}} = \frac{\alpha}{1-s} \end{aligned} \qquad (38)$$

It should be noted that this approximation has been used successfully in other references, including, for example,.[89] For the p-spin symmetry, we now consider Eq. (31) and will follow similar steps to the spin-symmetry case in the deformation of Dirac theory symmetries. Same as before, Eq. (31) cannot be solved exactly for $k = 0$ and $k = 1$ without tensor interaction, since the two centrifugal terms (proportional to $k(k-1)\, r^{-2}$and $k(k-1)\, r^{-4}$). Applying the approximations Eq. (37) to the centrifugal terms of Eqs. (35) and (36), the general form of the additive potentials $\Sigma_{\mathrm{my}}^{\mathrm{pert}}(r)$ and $\Delta_{\mathrm{my}}^{\mathrm{pert}}(r)$ will be as follows:

$$\sum_{my}^{pert}(r)=\Bigg(\beta_{nk}^{1s}\frac{1}{(1-s)^4}+\beta_{nk}^{2s}\frac{s}{(1-s)^3}+\beta_{nk}^{3s}\frac{s}{(1-s)^4}+\beta_{nk}^{4s}\frac{s}{(1-s)^2}+\beta_{nk}^{5s}\frac{s^2}{(1-s)^3}+\beta_{nk}^{6s}\frac{s^2}{(1-s)^4}+\beta_{nk}^{7s}\frac{s}{(1-s)^4}+\beta_{nk}^{8s}\frac{s^2}{(1-s)^3}+\beta_{nk}^{9s}\frac{s^2}{(1-s)^4}+\beta_{nk}^{10s}\frac{s^3}{(1-s)^4}\Bigg)\mathbf{L}\boldsymbol{\eta} \tag{39}$$

and

$$\Delta_{my}^{pert}(r)=\Bigg(\beta_{nk}^{1ps}\frac{1}{(1-s)^4}+\beta_{nk}^{2ps}\frac{s}{(1-s)^3}+\beta_{nk}^{3ps}\frac{s}{(1-s)^4}+\beta_{nk}^{4ps}\frac{s}{(1-s)^2}+\beta_{nk}^{5ps}\frac{s^2}{(1-s)^3}+\beta_{nk}^{6ps}\frac{s^2}{(1-s)^4}+\beta_{nk}^{7ps}\frac{s}{(1-s)^4}+\beta_{nk}^{8ps}\frac{s^2}{(1-s)^3}+\beta_{nk}^{9ps}\frac{s^2}{(1-s)^4}+\beta_{nk}^{10ps}\frac{s^3}{(1-s)^4}\Bigg)\mathbf{L}^p\boldsymbol{\eta} \tag{40}$$

with

$$\begin{cases}\beta_{\mathrm{nk}}^{1s}=\left(k(k+1)+a_1^-\right)\alpha^4,\\ \beta_{\mathrm{nk}}^{2s}=-\frac{\left(a_7+B\right)\alpha^3}{2}-\frac{\alpha^3a_3}{2}\\ \beta_{\mathrm{nk}}^{3s}=a_2\alpha^4+\frac{\alpha^4a_3}{2},\\ \beta_{\mathrm{nk}}^{4s}=\alpha^2\left(\frac{\alpha b_1^--\alpha B}{2}+\frac{\alpha^2a_5^-}{2}\right)\\ \beta_{\mathrm{nk}}^{5s}=\alpha b_2\alpha^3+\alpha A\alpha^3+\frac{\alpha^2a_5^{\mp}}{2},\\ \beta_{\mathrm{nk}}^{6s}=\left(b_2+A\right)\alpha^4+\frac{\alpha^3a_3}{2}\\ \beta_{\mathrm{nk}}^{7s}=\frac{\alpha^4a_3}{2},\beta_{\mathrm{nk}}^{8s}=\alpha^2a_4^-+\alpha^3a_6,\\ \beta_{\mathrm{nk}}^{9s}=\frac{\alpha^3a_6}{2}\text{ and }\beta_{\mathrm{nk}}^{10s}=\frac{\alpha^3a_6}{2}+\alpha^2a_4^-\end{cases} \tag{41.1}$$

while

$$\begin{cases}\beta_{\mathrm{nk}}^{1\mathrm{ps}}=\left(k(k-1)+a_1^+\right)\alpha^4,\\ \beta_{\mathrm{nk}}^{2\mathrm{ps}}=-\frac{\left(a_7+B\right)\alpha^3}{2}-\frac{\alpha^3a_3}{2}\\ \beta_{\mathrm{nk}}^{3\mathrm{ps}}=a_2\alpha^4+\frac{\alpha^4a_3}{2},\\ \beta_{\mathrm{nk}}^{4\mathrm{ps}}=\alpha^2\left(\frac{\alpha b_1^+-\alpha B}{2}+\frac{\alpha^2a_5^-}{2}\right)\\ \beta_{\mathrm{nk}}^{5\mathrm{sp}}=\alpha b_2\alpha^3+\alpha A\alpha^3+\frac{\alpha^2a_5^+}{2},*\beta_{\mathrm{nk}}^{6\mathrm{ps}}=\left(b_2+A\right)\alpha^4+\frac{\alpha^3a_3}{2}\\ \beta_{\mathrm{nk}}^{7\mathrm{ps}}=\frac{\alpha^4a_3}{2},\beta_{\mathrm{nk}}^{8\mathrm{ps}}=\alpha^2a_4^++\alpha^3a_6,\\ \beta_{\mathrm{nk}}^{9\mathrm{ps}}=\frac{\alpha^3a_6}{2}\text{ and }\beta_{\mathrm{nk}}^{10\mathrm{ps}}=\frac{\alpha^3a_6}{2}+\alpha^2a_4^+\end{cases} \tag{41.2}$$

In particular, the results obtained by the Greene and Aldrich approximation, for small values of $\alpha r<<1$, are in good agreement with those obtained using other methods. We have replaced the term $k(k+1)\,r^{-4}$ and $k(k-1)\,r^{-4}$ with the approximation in Eq. (37). The modified Yukawa potential under spin(pseudo) symmetries with three tensor interactions is extended by including new additive potentials $\Sigma_{\mathrm{my}}^{\mathrm{pert}}(r)$ and $\Delta_{\mathrm{my}}^{\mathrm{pert}}(r)$ expressed in terms proportional to the radial terms $\frac{1}{(1-s)^4}$, $\frac{s}{(1-s)^3}$, $\frac{s}{(1-s)^4}$, $\frac{s}{(1-s)^2}$, $\frac{s^2}{(1-s)^3}$, $\frac{s^2}{(1-s)^4}$, $\frac{s}{(1-s)^4}$, $\frac{s^2}{(1-s)^3}$, $\frac{s^2}{(1-s)^4}$ and $\frac{s^3}{(1-s)^4}$ to become the improved modified Yukawa potential under spin(pseudo) symmetries with three improved tensor interactions in deformation Dirac theory symmetries. The generated new two effective potentials $\Sigma_{\mathrm{my}}^{\mathrm{pert}}(r)$ and $\Delta_{\mathrm{my}}^{\mathrm{pert}}(r)$ are also proportional to the infinitesimal vector η. This allows us to consider the new additive parts of the effective potential $\Sigma_{\mathrm{my}}^{\mathrm{pert}}(r)$and $\Delta_{\mathrm{my}}^{\mathrm{pert}}(r)$ as perturbation potentials compared with the main potentials $\Sigma_{\mathrm{my}}(r)$ and $\Delta_{\mathrm{my}}(r)$, the parent potential operator in the symmetries of Dirac theory of deformation, that is, the two inequality$\Sigma_{\mathrm{my}}^{\mathrm{pert}}(r)<<\Sigma_{\mathrm{my}}(r)$ and $\Delta_{\mathrm{my}}^{\mathrm{pert}}(r)<<\Delta_{\mathrm{my}}(r)$ has become achieved to calculate the expectation values of previous radial terms. That is all physical justifications for applying the time-independent perturbation theory become satisfied. This allows us to give a complete prescription for determining the energy level of the generalized n^{th} excited states.

3.2. THE EXPECTATION VALUES UNDER NMYP IN THE DDT FOR SPIN SYMMETRY

In this subsection, we want to apply the perturbative theory, in the case of deformation Dirac theory symmetries, we find the expectation values$M_{1(\mathrm{nlm})}^{s-\mathrm{my}}\equiv\left\langle\frac{1}{(1-s)^4}\right\rangle_{(\mathrm{nlm})}^{s-\mathrm{my}}$, $M_{2(\mathrm{nlm})}^{s-\mathrm{my}}\equiv\left\langle\frac{s}{(1-s)^3}\right\rangle_{(\mathrm{nlm})}^{s-\mathrm{my}}$, $M_{3(\mathrm{nlm})}^{s-\mathrm{my}}\equiv\left\langle\frac{s}{(1-s)^4}\right\rangle_{(\mathrm{nlm})}^{s-\mathrm{my}}$, $M_{4(\mathrm{nlm})}^{s-\mathrm{my}}\equiv\left\langle\frac{s}{(1-s)^2}\right\rangle_{(\mathrm{nlm})}^{s-\mathrm{my}}$, $M_{5(\mathrm{nlm})}^{s-\mathrm{my}}\equiv\left\langle\frac{s^2}{(1-s)^3}\right\rangle_{(\mathrm{nlm})}^{s-\mathrm{my}}$, $M_{6(\mathrm{nlm})}^{s-\mathrm{my}}\left\langle\frac{s^2}{(1-s)^4}\right\rangle_{(\mathrm{nlm})}^{s-\mathrm{my}}$, $M_{7(\mathrm{nlm})}^{s-\mathrm{my}}\equiv\left\langle\frac{s}{(1-s)^4}\right\rangle_{(\mathrm{nlm})}^{s-\mathrm{my}}$, $M_{8(\mathrm{nlm})}^{s-\mathrm{my}}\equiv\left\langle\frac{s^2}{(1-s)^3}\right\rangle_{(\mathrm{nlm})}^{s-\mathrm{my}}$, $M_{9(\mathrm{nlm})}^{s-\mathrm{my}}\equiv M_{9(nim)}^{s-my}\equiv\left\langle\frac{s^2}{(1-s)^4}\right\rangle_{(nim)}^{s-my}$ and $M_{10(\mathrm{nlm})}^{s-\mathrm{my}}\equiv\left\langle\frac{s^3}{(1-s)^4}\right\rangle_{(\mathrm{nlm})}^{s-\mathrm{my}}$ for the spin symmetry taking into account the wave function which we have seen previously in Eq. (20). Thus, after straightforward calculations, we obtain the following results:

$$M_{1(\mathrm{nlm})}^{s-\mathrm{my}}=N_{\mathrm{nk}}^{\mathrm{ns2}}\int_0^{+\infty}s^{2\sqrt{W^s}}(1-s)^{2\chi_{\mathrm{nk}}^s-3}\Upsilon\left(n,\,\chi_{\mathrm{nk}}^s,\,W^s,\,s\right)dr \tag{42.1}$$

$$M_{2(\mathrm{nlm})}^{s-\mathrm{my}}=N_{\mathrm{nk}}^{\mathrm{ns2}}\int_0^{+\infty}s^{2\sqrt{W^s}+1}(1-s)^{2\chi_{\mathrm{nk}}^s-2}\Upsilon\left(n,\,\chi_{\mathrm{nk}}^s,\,W^s,\,s\right)dr \tag{42.2}$$

$$M_{3(\mathrm{nlm})}^{s-\mathrm{my}}=N_{\mathrm{nk}}^{\mathrm{ns2}}\int_0^{+\infty}s^{2\sqrt{W^s}+1}(1-s)^{2\chi_{\mathrm{nk}}^s-3}\Upsilon\left(n,\,\chi_{\mathrm{nk}}^s,\,W^s,\,s\right)dr \tag{42.3}$$

$$M_{4(\mathrm{nlm})}^{s-\mathrm{my}}=N_{\mathrm{nk}}^{\mathrm{ns2}}\int_0^{+\infty}s^{2\sqrt{W^s}+1}(1-s)^{2\chi_{\mathrm{nk}}^s-1}\Upsilon\left(n,\,\chi_{\mathrm{nk}}^s,\,W^s,\,s\right)dr \tag{42.4}$$

$$M_{5(\mathrm{nlm})}^{s-\mathrm{my}}=N_{\mathrm{nk}}^{\mathrm{ns2}}\int_0^{+\infty}s^{2\sqrt{W^s}+2}(1-s)^{2\chi_{\mathrm{nk}}^s-2}\Upsilon\left(n,\,\chi_{\mathrm{nk}}^s,\,W^s,\,s\right)dr \tag{42.5}$$

$$M_{6(\mathrm{nlm})}^{s-\mathrm{my}}=N_{\mathrm{nk}}^{\mathrm{ns2}}\int_0^{+\infty}s^{2\sqrt{W^s}+2}(1-s)^{2\chi_{\mathrm{nk}}^s-3}\Upsilon\left(n,\,\chi_{\mathrm{nk}}^s,\,W^s,\,s\right)dr \tag{42.6}$$

$$M_{7(\mathrm{nlm})}^{s-\mathrm{my}}=N_{\mathrm{nk}}^{\mathrm{ns2}}\int_0^{+\infty}s^{2\sqrt{W^s}+1}(1-s)^{2\chi_{\mathrm{nk}}^s-3}\Upsilon\left(n,\,\chi_{\mathrm{nk}}^s,\,W^s,\,s\right)dr \tag{42.7}$$

$$M_{8(\mathrm{nlm})}^{s-\mathrm{my}}=N_{\mathrm{nk}}^{\mathrm{ns2}}\int_0^{+\infty}s^{2\sqrt{W^s}+2}(1-s)^{2\chi_{\mathrm{nk}}^s-2}\Upsilon\left(n,\,\chi_{\mathrm{nk}}^s,\,W^s,\,s\right)dr \tag{42.8}$$

$$M_{9(\mathrm{nlm})}^{s-\mathrm{my}}=N_{\mathrm{nk}}^{\mathrm{ns2}}\int_0^{+\infty}s^{2\sqrt{W^s}+2}(1-s)^{2\chi_{\mathrm{nk}}^s-3}\Upsilon\left(n,\,\chi_{\mathrm{nk}}^s,\,W^s,\,s\right)dr \tag{42.9}$$

$$M_{10(\mathrm{nlm})}^{s-\mathrm{my}}=N_{\mathrm{nk}}^{\mathrm{ns2}}\int_0^{+\infty}s^{2\sqrt{W^s}+3}(1-s)^{2\chi_{\mathrm{nk}}^s-3}\Upsilon\left(n,\,\chi_{\mathrm{nk}}^s,\,W^s,\,s\right)dr \tag{42.10}$$

with $\Upsilon\left(n,\,\chi_{\mathrm{nk}}^s,\,W^s,\,s\right)\equiv\left[2F_1\left(-n,\,Q_{\mathrm{nk}}^s;\,V^s,\,s\right)\right]^2$. We have used useful abbreviations $\langle R\rangle_{(\mathrm{nlm})}^{s-\mathrm{my}}=\langle n,\,l,\,m\,|R|n,\,l,\,m\rangle$ to avoid the extra burden of writing equations. Furthermore, we have applied the property of the spherical harmonics, which has the form:

$$\int Y_l^m\left(\theta',\varphi'\right)Y_{l'}^{m'}(\theta,\varphi)\sin(\theta)d\theta d\varphi=\delta_{ll'}\delta_{mm'}$$

Introducing the change of variable $s=\exp(-\alpha r)$. This maps the region $0\leq r\prec\infty$ to $0\leq s\leq1$ and allows us to obtain $dr=-\frac{ds}{\alpha s}$, and transform Eqs.(42, $i=\overline{1,\,10}$) into the following form:

$$M^{s-\mathrm{my}}_{1(\mathrm{nlm})} = \frac{N^{\mathrm{ns2}}_{\mathrm{nk}}}{\alpha}\int_0^{+1} s^{2\sqrt{W^s}-1}(1-s)^{2\chi^s_{\mathrm{nk}}-2-1}\Upsilon\left(n,\ \chi^s_{\mathrm{nk}},\ W^s,\ s\right)ds \tag{43.1}$$

$$M^{s-\mathrm{my}}_{2(\mathrm{nlm})} = \frac{N^{\mathrm{ns2}}_{\mathrm{nk}}}{\alpha}\int_0^{+1} s^{2\sqrt{W^s}+1-1}(1-s)^{2\chi^s_{\mathrm{nk}}-1-1}\Upsilon\left(n,\ \chi^s_{\mathrm{nk}},\ W^s,\ s\right)ds \tag{43.2}$$

$$M^{s-\mathrm{my}}_{3(\mathrm{nlm})} = \frac{N^{\mathrm{ns2}}_{\mathrm{nk}}}{\alpha}\int_0^{+1} s^{2\sqrt{W^s}+1-1}(1-s)^{2\chi^s_{\mathrm{nk}}-2-1}\Upsilon\left(n,\ \chi^s_{\mathrm{nk}},\ W^s,\ s\right)ds \tag{43.3}$$

$$M^{s-\mathrm{my}}_{4(\mathrm{nlm})} = \frac{N^{\mathrm{ns2}}_{\mathrm{nk}}}{\alpha}\int_0^{+1} s^{2\sqrt{W^s}+1-1}(1-s)^{2\chi^s_{\mathrm{nk}}-1}\Upsilon\left(n,\ \chi^s_{\mathrm{nk}},\ W^s,\ s\right)ds \tag{43.4}$$

$$M^{s-\mathrm{my}}_{5(\mathrm{nlm})} = \frac{N^{\mathrm{ns2}}_{\mathrm{nk}}}{\alpha}\int_0^{+1} s^{2\sqrt{W^s}+2-1}(1-s)^{2\chi^s_{\mathrm{nk}}-1-1}\Upsilon\left(n,\ \chi^s_{\mathrm{nk}},\ W^s,\ s\right)ds \tag{43.5}$$

$$M^{s-\mathrm{my}}_{6(\mathrm{nlm})} = \frac{N^{\mathrm{ns2}}_{\mathrm{nk}}}{\alpha}\int_0^{+1} s^{2\sqrt{W^s}+2-1}(1-s)^{2\chi^s_{\mathrm{nk}}-2-1}\Upsilon\left(n,\ \chi^s_{\mathrm{nk}},\ W^s,\ s\right)ds \tag{43.6}$$

$$M^{s-\mathrm{my}}_{7(\mathrm{nlm})} = \frac{N^{\mathrm{ns2}}_{\mathrm{nk}}}{\alpha}\int_0^{+1} s^{2\sqrt{W^s}+1-1}(1-s)^{2\chi^s_{\mathrm{nk}}-2-1}\Upsilon\left(n,\ \chi^s_{\mathrm{nk}},\ W^s,\ s\right)ds \tag{43.7}$$

$$M^{s-\mathrm{my}}_{8(\mathrm{nlm})} = \frac{N^{\mathrm{ns2}}_{\mathrm{nk}}}{\alpha}\int_0^{+1} s^{2\sqrt{W^s}+2-1}(1-s)^{2\chi^s_{\mathrm{nk}}-1-1}\Upsilon\left(n,\ \chi^s_{\mathrm{nk}},\ W^s,\ s\right)ds \tag{43.8}$$

$$M^{s-\mathrm{my}}_{9(\mathrm{nlm})} = \frac{N^{\mathrm{ns2}}_{\mathrm{nk}}}{\alpha}\int_0^{+1} s^{2\sqrt{W^s}+2-1}(1-s)^{2\chi^s_{\mathrm{nk}}-2-1}\Upsilon\left(n,\ \chi^s_{\mathrm{nk}},\ W^s,\ s\right)ds \tag{43.9}$$

$$M^{s-\mathrm{my}}_{10(\mathrm{nlm})} = \frac{N^{\mathrm{ns2}}_{\mathrm{nk}}}{\alpha}\int_0^{+1} s^{2\sqrt{W^s}+3-1}(1-s)^{2\chi^s_{\mathrm{nk}}-2-1}\Upsilon\left(n,\ \chi^s_{\mathrm{nk}},\ W^s,\ s\right)ds \tag{43.10}$$

We can evaluate the above integrals either in a recurrence way through the physical values of the principal quantum number ($n = 0,\ 1, ...$) and then generalize the result to the general n^{th} excited state or we use the method proposed by Dong *et al.*[90] and applied by Zhang,[91] to obtain the general excited state directly. We calculate the integrals in Eqs.(43, $i = \overline{1,\ 10}$) with the help of the special integral formula:

$$\begin{aligned}&\int_0^{+1} s^{\alpha-1}(1-s)^{\beta-1}\left[{}_2F_1\left(c_1, c_2; c_3; s\right)\right]^2 ds\\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}{}_3F_2\left(c_1, c_2, \beta; c_3, \beta+\alpha; 1\right)\end{aligned} \tag{44}$$

here ${}_2F_1\left(c_1,\ c_2;\ c_3;\ s\right)$ is the generalized hypergeometric function and ${}_3F_2\left(c_1,\ c_2,\ \beta;\ c_3,\ \beta+\alpha;\ 1\right)$ which is equal $\sum_{n=0}^{+\infty}\frac{(c_1)_n(c_2)_n(\sigma)_n}{(c_3)_n(\sigma+\xi)\,n!}$, $(c_1)_n$ is the rising factorial or Pochhammer symbol while $\Gamma(\alpha)$ denoting the usual Gamma function. By identifying Eq. (44) with the integrals, we obtain the following results:

$$M^{s-\mathrm{my}}_{1(\mathrm{nlm})} = N^{\mathrm{ns2}}_{\mathrm{nk}}\frac{\Gamma\left(2\sqrt{W^s}\right)\Gamma\left(2\chi^s_{\mathrm{nk}}-2\right)}{\Gamma\left(X^s_{\mathrm{nk}}-2\right)}$$
$$3F_2\left(-n,\ Q^s_{\mathrm{nk}},\ 2\chi^s_{\mathrm{nk}}-2;\ V^s,\ X^s_{\mathrm{nk}}-2;\ 1\right) \tag{45.1}$$

$$M^{s-\mathrm{my}}_{2(\mathrm{nlm})} = N^{\mathrm{ns2}}_{\mathrm{nk}}\frac{\Gamma\left(2\sqrt{W^s}+1\right)\Gamma\left(2\chi^s_{\mathrm{nk}}-1\right)}{\Gamma\left(X^s_{\mathrm{nk}}\right)}$$
$$3F_2\left(-n,\ Q^s_{\mathrm{nk}},\ 2\chi^s_{\mathrm{nk}}-1;\ V^s,\ X^s_{\mathrm{nk}};\ 1\right) \tag{45.2}$$

$$M^{s-\mathrm{my}}_{3(\mathrm{nlm})} = N^{\mathrm{ns2}}_{\mathrm{nk}}\frac{\Gamma\left(2\sqrt{W^s}+1\right)\Gamma\left(2\chi^s_{\mathrm{nk}}-2\right)}{\Gamma\left(X^s_{\mathrm{nk}}-1\right)}$$
$$3F_2\left(-n,\ Q^s_{\mathrm{nk}},\ 2\chi^s_{\mathrm{nk}}-2;\ V^s,\ X^s_{\mathrm{nk}}-1;\ 1\right) \tag{45.3}$$

$$M^{s-\mathrm{my}}_{4(\mathrm{nlm})} = N^{\mathrm{ns2}}_{\mathrm{nk}}\frac{\Gamma\left(2\sqrt{W^s}+1\right)\Gamma\left(2\chi^s_{\mathrm{nk}}\right)}{\Gamma\left(X^s_{\mathrm{nk}}+1\right)}$$
$$3F_2\left(-n,\ Q^s_{\mathrm{nk}},\ 2\chi^s_{\mathrm{nk}};\ V^s,\ X^s_{\mathrm{nk}}+1;\ 1\right) \tag{45.4}$$

$$M^{s-\mathrm{my}}_{5(\mathrm{nlm})} = N^{\mathrm{ns2}}_{\mathrm{nk}}\frac{\Gamma\left(2\sqrt{W^s}+2\right)\Gamma\left(2\chi^s_{\mathrm{nk}}-1\right)}{\Gamma\left(X^s_{\mathrm{nk}}+1\right)}$$
$$3F_2\left(-n,\ Q^s_{\mathrm{nk}},\ 2\chi^s_{\mathrm{nk}}-1;\ V^s,\ X^s_{\mathrm{nk}}+1;\ 1\right) \tag{45.5}$$

$$M^{s-\mathrm{my}}_{6(\mathrm{nlm})} = N^{\mathrm{ns2}}_{\mathrm{nk}}\frac{\Gamma\left(2\sqrt{W^s}+2\right)\Gamma\left(2\chi^s_{\mathrm{nk}}-2\right)}{\Gamma\left(X^s_{\mathrm{nk}}\right)}$$
$$3F_2\left(-n,\ Q^s_{\mathrm{nk}},\ 2\chi^s_{\mathrm{nk}}-2;\ V^s,\ X^s_{\mathrm{nk}};\ 1\right) \tag{45.6}$$

$$M^{s-\mathrm{my}}_{7(\mathrm{nlm})} = \frac{N^{\mathrm{ns2}}_{\mathrm{nk}}}{\alpha}\frac{\Gamma\left(2\sqrt{W^s}+1\right)\Gamma\left(2\chi^s_{\mathrm{nk}}-2\right)}{\Gamma\left(X^s_{\mathrm{nk}}-1\right)}$$
$$3F_2\left(-n,\ Q^s_{\mathrm{nk}},\ \chi^s_{\mathrm{nk}}-2;\ V^s,\ X^s_{\mathrm{nk}}-1;\ 1\right) \tag{45.7}$$

$$M^{s-\mathrm{my}}_{8(\mathrm{nlm})} = \frac{N^{\mathrm{ns2}}_{\mathrm{nk}}}{\alpha}\frac{\Gamma\left(2\sqrt{W^s}+2\right)\Gamma\left(2\chi^s_{\mathrm{nk}}-1\right)}{\Gamma\left(X^s_{\mathrm{nk}}+1\right)}$$
$$3F_2\left(-n,\ Q^s_{\mathrm{nk}},\ 2\chi^s_{\mathrm{nk}}+1;\ V^s,\ 2X^s_{\mathrm{nk}}-1;\ 1\right) \tag{45.8}$$

$$M^{s-\mathrm{my}}_{9(\mathrm{nlm})} = \frac{N^{\mathrm{ns2}}_{\mathrm{nk}}}{\alpha}\frac{\Gamma\left(2\sqrt{W^s}+2\right)\Gamma\left(2\chi^s_{\mathrm{nk}}-2\right)}{\Gamma\left(X^s_{\mathrm{nk}}\right)}$$
$$3F_2\left(-n,\ Q^s_{\mathrm{nk}},\ 2\chi^s_{\mathrm{nk}};\ V^s,\ X^s_{\mathrm{nk}};\ 1\right) \tag{45.9}$$

and

$$M^{s-\mathrm{my}}_{10(\mathrm{nlm})} = N^{\mathrm{ns2}}_{\mathrm{nk}}\frac{\Gamma\left(2\sqrt{W^s}+3\right)\Gamma\left(2\chi^s_{\mathrm{nk}}-2\right)}{\Gamma\left(X^s_{\mathrm{nk}}+1\right)}$$
$$3F_2\left(-n,\ Q^s_{\mathrm{nk}},\ 2\chi^s_{\mathrm{nk}}-2;\ V^s,\ X^s_{\mathrm{nk}}+1;\ 1\right) \tag{45.10}$$

with $X^s_{\mathrm{nk}} = 2\sqrt{W^s} + 2\chi^s_{\mathrm{nk}}$.

3.3. THE EXPECTATION VALUES UNDER NMYP IN THE DDT FOR THE P-SPIN SYMMETRY

Now, we apply the perturbative theory to find the following expectation values $M^{s-\mathrm{my}}_{1(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}\left\langle\frac{1}{(1-s)^4}\right\rangle^{s-\mathrm{my}}_{(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}$, $M^{\mathrm{ps-my}}_{2(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}\left\langle\frac{s}{(1-s)^3}\right\rangle^{\mathrm{ps-my}}_{(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}$, $M^{\mathrm{ps-my}}_{3(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}\left\langle\frac{s}{(1-s)^4}\right\rangle^{\mathrm{ps-my}}_{(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}$, $M^{\mathrm{ps-my}}_{4(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}\left\langle\frac{s}{(1-s)^2}\right\rangle^{\mathrm{ps-my}}_{(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}$, $M^{\mathrm{ps-my}}_{5(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}\left\langle\frac{s^2}{(1-s)^3}\right\rangle^{\mathrm{ps-my}}_{(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}$, $M^{\mathrm{ps-my}}_{6(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}\left\langle\frac{s^2}{(1-s)^4}\right\rangle^{\mathrm{ps-my}}_{(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}$, $M^{\mathrm{ps-my}}_{7(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}\left\langle\frac{s}{(1-s)^4}\right\rangle^{\mathrm{ps-my}}_{(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}$, $M^{\mathrm{ps-my}}_{8(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}\left\langle\frac{s^2}{(1-s)^3}\right\rangle^{\mathrm{ps-my}}_{(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}$, $M^{\mathrm{ps-my}}_{9(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}\left\langle\frac{s^2}{(1-s)^4}\right\rangle^{\mathrm{ps-my}}_{(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}$ and $M^{\mathrm{ps-my}}_{10(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}\left\langle\frac{s^3}{(1-s)^4}\right\rangle^{\mathrm{ps-my}}_{(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}$ for p-spin symmetry under NMYP in the DDT with three tensor interactions taking into account the wave function which we have seen previously in Eq. (21). We examine the two expressions of the two wave functions shown in Eqs. (20) and (21), we note that there is a possibility of moving from the upper wave function $F_{\mathrm{nk}}(r)$ to the other lower wave function $G_{\mathrm{nk}}(r)$ by making the following substitutions:

$$N^s_{\mathrm{nk}} \Leftrightarrow N^{\mathrm{ps}}_{\mathrm{nk}}, \sqrt{W^s} \Leftrightarrow \sqrt{W^{\mathrm{ps}}} \text{ and } \chi^s_{\mathrm{nk}} \Leftrightarrow \chi^{\mathrm{ps}}_{\mathrm{nk}} \tag{46}$$

allows us to obtain the expectation values for p-spin symmetry from Eqs.(45, $i = \overline{1,\ 10}$) without re-calculation, as follows:

$$M^{ps-my}_{1(nl^{ps}m^{ps})} = N^{nps2}_{nk}\frac{\Gamma\left(2\sqrt{W^{ps}}\right)\Gamma\left(2\chi^{ps}_{nk}-2\right)}{\Gamma\left(X^{ps}_{nk}-2\right)}$$
$${}_3F_2\left(-n,\ Q^{ps}_{nk},\ 2\chi^{ps}_{nk}-2;\ V^{ps},\ X^{ps}_{nk}-2;\ 1\right) \tag{47.1}$$

$$M^{ps-my}_{2(nl^{ps}m^{ps})} = N^{nps2}_{nk}\frac{\Gamma\left(2\sqrt{W^{ps}}+1\right)\Gamma\left(2\chi^{ps}_{nk}-1\right)}{\Gamma\left(X^{ps}_{nk}\right)}$$
$${}_3F_2\left(-n,\ Q^{ps}_{nk},\ 2\chi^{ps}_{nk}-1;\ V^{ps},\ X^{ps}_{nk};\ 1\right) \tag{47.2}$$

$$M^{ps-my}_{3(nl^{ps}m^{ps})} = N^{nps2}_{nk}\frac{\Gamma\left(2\sqrt{W^{ps}}+1\right)\Gamma\left(2\chi^{ps}_{nk}-2\right)}{\Gamma\left(X^{ps}_{nk}-1\right)}$$
$${}_3F_2\left(-n,\ Q^{ps}_{nk},\ 2\chi^{ps}_{nk}-2;\ V^{ps},\ X^{ps}_{nk}-1;\ 1\right) \tag{47.3}$$

$$M^{ps-my}_{4(nl^{ps}m^{ps})}=N^{nps2}_{nk}\frac{\Gamma\left(2\sqrt{W^{ps}}+1\right)\Gamma(2\chi^{ps}_{nk})}{\Gamma(X^{ps}_{nk}+1)} \tag{47.4}$$
$${}_3F_2\left(-n,\ Q^{ps}_{nk},\ 2\chi^{ps}_{nk};\ V^{ps},\ X^{ps}_{nk}+1;\ 1\right)$$
$$M^{ps-my}_{5(nl^{ps}m^{ps})}=N^{nps2}_{nk}\frac{\Gamma\left(2\sqrt{W^{ps}}+2\right)\Gamma(2\chi^{ps}_{nk}-1)}{\Gamma(X^{ps}_{nk}+1)} \tag{47.5}$$
$${}_3F_2\left(-n,\ Q^{ps}_{nk},\ 2\chi^{ps}_{nk}-1;\ V^{ps},\ X^{ps}_{nk}+1;\ 1\right)$$
$$M^{ps-my}_{6(nl^{ps}m^{ps})}=N^{nps2}_{nk}\frac{\Gamma\left(2\sqrt{W^{ps}}+2\right)\Gamma(2\chi^{ps}_{nk}-2)}{\Gamma(X^{ps}_{nk})} \tag{47.6}$$
$${}_3F_2\left(-n,\ Q^{ps}_{nk},\ 2\chi^{ps}_{nk}-2;\ V^{ps},\ X^{ps}_{nk};\ 1\right)$$
$$M^{ps-my}_{7(nl^{ps}m^{ps})}=N^{nps2}_{nk}\frac{\Gamma\left(2\sqrt{W^{ps}}+1\right)\Gamma(\chi^{ps}_{nk}-2)}{\Gamma(X^{ps}_{nk}-1)} \tag{47.7}$$
$${}_3F_2\left(-n,\ Q^{ps}_{nk},\ \chi^{ps}_{nk}-2;\ V^{ps},\ X^{ps}_{nk}-1;\ 1\right)$$
$$M^{ps-my}_{8(nl^{ps}m^{ps})}=N^{nps2}_{nk}\frac{\Gamma\left(2\sqrt{W^{ps}}+2\right)\Gamma(2\chi^{ps}_{nk}-1)}{\Gamma(X^{ps}_{nk}+1)} \tag{47.8}$$
$${}_3F_2\left(-n,\ Q^{ps}_{nk},\ 2\chi^{ps}_{nk}-1;\ V^{ps},\ 2X^{ps}_{nk}+1;\ 1\right)$$
$$M^{ps-my}_{9(nl^{ps}m^{ps})}=N^{nps2}_{nk}\frac{\Gamma\left(2\sqrt{W^{ps}}+5\right)\Gamma(2\chi^{ps}_{nk}-2)}{\Gamma(X^{ps}_{nk})} \tag{47.9}$$
$${}_3F_2\left(-n,\ Q^{ps}_{nk},\ 2\chi^{ps}_{nk}-2;\ V^{ps},\ X^{ps}_{nk};\ 1\right)$$

and

$$M^{ps-my}_{10(nl^{ps}m^{ps})}=N^{nps2}_{nk}\frac{\Gamma\left(2\sqrt{W^{ps}}+3\right)\Gamma(2\chi^{ps}_{nk}-2)}{\Gamma(X^{ps}_{nk}+1)} \tag{47.10}$$
$${}_3F_2\left(-n,\ Q^{ps}_{nk},\ 2\chi^{ps}_{nk}-2;\ V^{ps},\ X^{ps}_{nk}+1;\ 1\right)$$

with $X^{ps}_{\mathrm{nk}}=2\sqrt{W^{ps}}+2\chi^{ps}_{nk}$.

3.4 THE CORRECTED ENERGY FOR THE NMYP IN DDT SYMMETRIES

Based on our strategy that we have successfully applied in previous works and which we try to develop in every new work. We can say that the global relativistic energy in the perspective of DDT produced with the NMYP model as a result of a major contribution to relativistic energy known in the literature under modified Yukawa potential under three-tensor interactions model in usual Dirac theory and which we paved for through a quick look for the spin(p-spin)-symmetry in Eqs. (17) and (18), while the new contributions are produced from the topological properties under space-space deformation, which can be evaluated through several contributions; we will address three of them. The first one has generated from the effect of the perturbed spin-orbit and pseudo(spin-orbit) effective potentials $\Sigma^{\mathrm{pert}}_{\mathrm{my}}(r)$ and $\Delta^{\mathrm{pert}}_{\mathrm{my}}(r)$ corresponds to spin symmetry and pseudospin symmetry. These perturbed effective potentials are obtained by replacing the coupling of the angular momentums ($\mathbf{L}$ and $\mathbf{L}^{\mathrm{ps}}$) operators and the NC vector η with the new equivalent couplings ηLS and $\eta\mathbf{L}^{\mathrm{ps}}\mathbf{S}^{\mathrm{ps}}$ for spin-symmetry and p-spin-symmetry, respectively (with $\eta^2=\eta^2_{12}+\eta^2_{23}+\eta^2_{13}$). This degree of freedom comes considering that the infinitesimal NC vector η is arbitrary. We have oriented the spins-$\left(\mathbf{S},\mathbf{S}^{\mathrm{ps2}}\right)$ of the fermionic particles to become parallels to the vector η which interacted with improved modified Yukawa potential including generalized (Coulomb-Yukawa and Hulthén)-like tensor interactions. Moreover, we replace the new spin-orbit and pseudo(spin-orbit) couplings ηLS and $\eta\mathbf{L}^{\mathrm{ps}}\mathbf{S}^{\mathrm{ps}}$ with the corresponding new physical form $\left(\frac{\eta}{2}\right)\mathbf{G}^2$ and $\left(\frac{\eta}{2}\right)\mathbf{G}^{\mathrm{ps2}}$, with $\mathbf{G}^2=\mathbf{J}^2-\mathbf{L}^2-\mathbf{S}^2$ and $\mathbf{G}^{\mathrm{ps2}}=\mathbf{J}^2-\mathbf{L}^{\mathrm{ps2}}-\mathbf{S}^{\mathrm{ps2}}$ for a spin (p-spin)-symmetry, respectively. Furthermore, in RQM, the operators ($H^{\mathrm{my}}_{\mathrm{nc}-D}$, $?\mathbf{J}^2$, $\mathbf{L}^2$, $\mathbf{S}^2$ and $\mathbf{J}_z$) form a complete set of conserved physics quantities, the eigenvalues of the operators $\mathbf{G}^2$ and $\mathbf{G}^{\mathrm{ps2}}$ are equal to the values $\chi(j,\ l,\ s)=\frac{[j(j+1)-k(k+1)-\frac{3}{4})]}{2}$ and $\chi\left(j,\ l^{\mathrm{ps}},\ s^{\mathrm{ps}}\right)=\frac{[j(j+1)-k(k-1)-\frac{3}{4})]}{2}$, with $\left|l-\frac{1}{2}\right|\ j\leq\ \left|l+\frac{1}{2}\right|$ and $\left|l^{\mathrm{ps}}-\frac{1}{2}\right|\ j\leq\ \left|l^{\mathrm{ps}}+\frac{1}{2}\right|$ for spin-symmetry and p-spin-symmetry, respectively. As a direct consequence, the partially corrected energies $\Delta E^{\mathrm{so}-s}_{\mathrm{my}}(n,\ \alpha,\ A,\ B,\ H_c,\ H_Y,\ H_H,\ \eta,\ j,\ l,\ s)$ $\Delta E^{\mathrm{so}-s}_{\mathrm{my}}$ and $\Delta E^{\mathrm{so}-\mathrm{ps}}_{\mathrm{my}}(n,\ \alpha,\ A,\ B,\ H_c,\ H_Y,\ H_H,\ \eta,\ j,\ l^{\mathrm{ps}},\ s^{\mathrm{ps}})$ $\Delta E^{\mathrm{so}-\mathrm{ps}}_{\mathrm{my}}$ due to the perturbed effective potentials $\Sigma^{\mathrm{pert}}_{\mathrm{my}}(r)$ and $\Delta^{\mathrm{pert}}_{\mathrm{my}}(r)$ produced for the nth excited state, in DDT symmetries are as follows:

$$\begin{cases}\Delta E^{\mathrm{so}-s}_{\mathrm{my}}=\eta\left(j(j+1)-k(k+1)-\frac{3}{4})\right)\\ \langle V\rangle^{\mathrm{my}}_{(\mathrm{nlm})}(n,\ \alpha,\ A,\ B,\ H_c,\ H_Y,\ H_H)\\ \Delta E^{\mathrm{so}-\mathrm{ps}}_{\mathrm{my}}=\eta\left(j(j+1)-k(k-1)-\frac{3}{4}\right)\\ \langle V^{\mathrm{ps}}\rangle^{\mathrm{my}}_{(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}(n,\ \alpha,\ A,\ B,\ H_c,\ H_Y,\ H_H)\end{cases} \tag{48}$$

The global two expectation values $\langle V\rangle^{\mathrm{my}}_{(\mathrm{nlm})}(n,\ \alpha,\ A,\ B,\ H_c,\ H_Y,\ H_H)$ and $\langle V^{\mathrm{ps}}\rangle^{\mathrm{my}}_{(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}(n,\ \alpha,\ A,\ B,\ H_c,\ H_Y,\ H_H)$ for a spin (p-spin)-symmetry, respectively are determined from the following expressions:

$$\begin{cases}\langle V\rangle^{my}_{(nlm)}(n,\alpha,A,B,H_c,H_Y,H_H)=\sum^{10}_{\mu=1}\beta^{\mu s}_{nk}M^{s-my}_{\mu(nlm)}\\ \langle V^{ps}\rangle^{my}_{(nl^{ps}m^{ps})}(n,\alpha,A,B,H_c,H_Y,H_H)=\sum^{10}_{\mu=1}\beta^{\mu ps}_{nk}M^{ps-my}_{\mu(nl^{ps}m^{ps})}\end{cases} \tag{49}$$

Where $\beta^{\mu s}_{\mathrm{nk}}$ and$\beta^{\mu\mathrm{ps}}_{\mathrm{nk}}\left(\mu=\overline{1,\ 10}\right)$ are determined from Eqs. (40) and (41) while$M^{s-\mathrm{my}}_{\mu(\mathrm{nlm})}$ and $M^{\mathrm{ps}-\mathrm{my}}_{\mu(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}$ are determined from Eqs. (45, $i=\overline{1,\ 10}$) and Eqs. (47, $i=\overline{1,\ 10}$), respectively. The second main part is obtained from the magnetic effect of the perturbative effective potentials $\Sigma^{\mathrm{pert}}_{\mathrm{my}}(r)$ and $\Delta^{\mathrm{pert}}_{\mathrm{my}}(r)$ under the NMYP model in the deformation of Dirac theory symmetries. These effective potentials are achieved when we replace both ($\mathbf{L}\eta$ and $\mathbf{L}^{\mathrm{ps}}\eta$) by ($\sigma\aleph L_z$ and $\sigma\aleph l^{\mathrm{ps}}_z$), respectively, and η_{12} by $\sigma\aleph$ and η_{12} by $\sigma\aleph$, here ($\aleph$ and σ) are present the intensity of the magnetic field induced by the effect of the deformation of space-space geometry and a new infinitesimal noncommutativity parameter, so that the physical unit of the original noncommutativity parameter η_{12} (length)^2 is the same unit of $\sigma\aleph$, we have also need to apply $\langle n^{'},\ l^{'},\ m^{'}\ |L_z|n,\ l,\ m\rangle=m\delta_{m^{'}m}\delta_{l^{'}l}\delta_{n^{'}n}$ and $\langle n^{'},\ l^{\mathrm{ps}},\ m^{\mathrm{ps}}\ \left|L^{\mathrm{ps}}_z\right|n,\ l^{\mathrm{ps}},\ m^{\mathrm{ps}}\rangle=m^{\mathrm{ps}}\delta_{m^{\mathrm{ps}}m^{\mathrm{ps}}}\delta_{l^{\mathrm{ps}}l^{\mathrm{ps}}}\delta_{n^{'}n}$ ($-l^{\mathrm{ps}}\leq m^{\mathrm{ps}}\leq\ l^{\mathrm{ps}}$and $-l\leq m\leq\ l$) for a spin(p-spin)-symmetry, respectively. All of these data allow for the discovery of the new energy shift $\Delta E^{\mathrm{mg}-s}_{\mathrm{my}}(n,\ \alpha,\ A,\ B,\ H_c,\ H_Y,\ H_H,\ \sigma,\ m)$ and $\Delta E^{\mathrm{mg}-\mathrm{ps}}_{\mathrm{my}}(n,\ \alpha,\ A,\ B,\ H_c,\ H_Y,\ H_H,\ \sigma,\ m^{\mathrm{ps}})$ due to the perturbed Zeeman effect created by the influence of the NMYP model for the nth excited state in deformation Dirac theory symmetries as follows:

$$\begin{cases}\Delta E^{\mathrm{mg}-s}_{\mathrm{my}}(n,\ \alpha,\ A,\ B,\ H_c,\ H_Y,\ H_H,\ \sigma,\ m)=\sigma\aleph\\ \langle V\rangle^{\mathrm{my}}_{(\mathrm{nlm})}(n,\ \alpha,\ A,\ B,\ H_c,\ H_Y,\ H_H)\ m\\ \Delta E^{\mathrm{mg}-\mathrm{ps}}_{\mathrm{my}}(n,\ \alpha,\ A,\ B,\ H_c,\ H_Y,\ H_H,\ \sigma,\ m^{\mathrm{ps}})=\sigma\aleph\\ \langle V^{\mathrm{ps}}\rangle^{\mathrm{my}}_{(\mathrm{nlm}^{\mathrm{ps}})}(n,\ \alpha,\ A,\ B,\ H_c,\ H_Y,\ H_H)\ m^{\mathrm{ps}}\end{cases} \tag{50}$$

We will now refer to the generation of another phenomenon as a result of the influence of topological properties under the NMYP model on DDT symmetries. This physical phenomenon is produced automatically from the influence of perturbed effective potentials $\Sigma^{\mathrm{pert}}_{\mathrm{my}}(r)$ and $\Delta^{\mathrm{pert}}_{\mathrm{my}}(r)$ which

we have seen in Eqs. (39) and (40). We consider the fermionic particles undergoing rotation with angular velocity $\mathbf{\Omega}$. The features of this subjective phenomenon are determined through the replace the arbitrary vector η with $\chi\mathbf{\Omega}$. Allowing us to replace the two couplings ($L\eta$ and $L^{\mathrm{ps}}\eta$) with ($\chi L\mathbf{\Omega}$ and $\chi L^{\mathrm{ps}}\mathbf{\Omega}$), respectively, as follows:

$$\begin{pmatrix} L\eta \\ L^{\mathrm{ps}}\eta \end{pmatrix} \rightarrow \chi \begin{pmatrix} \mathbf{\Omega L} \\ \mathbf{\Omega L}^{\mathrm{ps}} \end{pmatrix} \begin{cases} \text{for spin-sy.} \\ \text{for p-spin-sy.} \end{cases} \tag{51}$$

Here χ is just an infinitesimal real proportional constant. We can express the effective potential $\Sigma_{\mathrm{pert}}^{\mathrm{my-rot}}(s)$ and $\Delta_{\mathrm{pert}}^{\mathrm{my-rot}}(s)$ which induced the rotational movements of the fermionic particles as follows:

$$\begin{pmatrix} \sum_{my}^{\mathrm{pert}}(s) \\ \Delta_{my}^{\mathrm{pert}}(s) \end{pmatrix} \rightarrow \begin{pmatrix} \sum_{\mathrm{pert}}^{my-rot}(s) \\ \Delta_{\mathrm{pert}}^{my-rot}(s) \end{pmatrix} = \chi \begin{pmatrix} \left(\sum_{\mu=1}^{10} \beta_{nk}^{\mu s} M_{\mu(nlm)}^{s-my}\right)\mathbf{L} \\ \left(\sum_{\mu=1}^{10} \beta_{nk}^{\mu s} M_{\mu(nlm)}^{ps-my}\right)\mathbf{L}^{ps} \end{pmatrix} \mathbf{\Omega} \tag{52}$$

To simplify the calculations, we choose the rotational velocity $\mathbf{\Omega}$ parallel to the (Oz) axis ($\mathbf{\Omega} = \mathbf{\Omega}e_z$), this, of course, does not change the physical nature of the studied problem as much as it simplifies the calculations. Then we transform the spin-orbit couplings into the new physical phenomena as follows:

$$\begin{pmatrix} \Sigma_{\mathrm{pert}}^{\mathrm{my-rot}}(s)\,\mathbf{L} \\ \Delta_{\mathrm{pert}}^{\mathrm{my-rot}}(s)\,L^{\mathrm{ps}} \end{pmatrix} \mathbf{\Omega} = \chi\mathbf{\Omega} \begin{pmatrix} \Sigma_{\mathrm{pert}}^{\mathrm{my-rot}}(s)\,L_z \\ \Delta_{\mathrm{pert}}^{\mathrm{my-rot}}(s)\,l_z^{\mathrm{ps}} \end{pmatrix} \tag{53}$$

All of this data allows for the discovery of the new corrected energy $\Delta E_{\mathrm{my}}^{\mathrm{rot}-s}(n,\alpha,A,B,H_c,H_Y,H_H,\chi,m)$ and $\Delta E_{\mathrm{my}}^{\mathrm{rot-ps}}(n,\alpha,A,B,H_c,H_Y,H_H,\chi,m^{\mathrm{ps}})$ due to the perturbed effective potentials $\Sigma_{\mathrm{pert}}^{\mathrm{my-rot}}(s)$ and $\Delta_{\mathrm{pert}}^{\mathrm{my-rot}}(s)$ which is generated automatically by the influence of the new modified Yukawa potential including generalized (Coulomb-Yukawa and Hulthén)-like tensor interactions for the nth excited state in DDT symmetries as follows:

$$\begin{pmatrix} \Delta E_{\mathrm{my}}^{\mathrm{rot}-s} \\ \Delta E_{\mathrm{my}}^{\mathrm{rot-ps}} \end{pmatrix} = \chi\Omega \begin{pmatrix} \langle V\rangle_{(\mathrm{nlm})}^{\mathrm{my}} m \\ \langle V^{\mathrm{ps}}\rangle_{(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}^{\mathrm{my}} m^{\mathrm{ps}} \end{pmatrix} \tag{54}$$

It is worth mentioning that the authors of Ref.[92] studied rotating isotropic and anisotropic harmonically confined ultra-cold Fermi gas in a two and three-dimensional space at zero temperature, but in this study, the rotational term was added to the Hamiltonian operator, in contrast to our case, in which in our recent study the two rotation operators $\Sigma_{\mathrm{pert}}^{\mathrm{my-rot}}(s)\,\mathbf{L\Omega}$ and $\Delta_{\mathrm{pert}}^{\mathrm{my-rot}}(s)\,\mathbf{L}^{\mathrm{ps}}\mathbf{\Omega}$ automatically appear due to the augmented symmetries resulting from the deformation of space-space under the improved modified Yukawa potential including generalized tensor interactions (Coulomb-Yukawa and Hulthén)-like tensor interactions. We have seen that the eigenvalues $\chi(j,\,l,\,s)$ and $\chi\left(j,\,l^{\mathrm{ps}},\,s^{\mathrm{ps}}\right)$ of the operators ($\mathbf{G}^2$and $\mathbf{G}^{\mathrm{ps}2}$) are, respectively, equal to the values:

$$\begin{cases} \chi(j,\,l,\,s) = \frac{\left[j(j+1)-l(l+1)-\frac{3}{4}\right]}{2} \\ \chi\left(j,\,l^{\mathrm{ps}},\,s^{\mathrm{ps}}\right) = \frac{\left[j(j+1)-l^{\mathrm{ps}}(l^{\mathrm{ps}}-1)-\frac{3}{4}\right]}{2} \end{cases} \tag{55}$$

Thus, for the case of spin-1/2 fields corresponding to up polarity and Down polarity, the possible values of j are $l\pm\frac{1}{2}$ and $l^{\mathrm{ps}}\pm\frac{1}{2}$ for spin symmetry $\chi(j,\,l,\,s)$ and pseudospin symmetry $\chi\left(j,\,l^{\mathrm{ps}},\,s^{\mathrm{ps}}\right)$, are as follows:

$$\begin{aligned} &\chi\left(j=l\pm\tfrac{1}{2},\,s=\tfrac{1}{2}\right) \\ &=\tfrac{1}{2}\begin{cases} l \text{ Up polarity: } j=l+\frac{1}{2} \\ -(l+1)\ \text{ Dawn polarity: } j=l-\frac{1}{2} \end{cases} \end{aligned} \tag{56.1}$$

and

$$\begin{aligned} &\chi\left(j=l^{\mathrm{ps}}\pm\tfrac{1}{2},\,s^{\mathrm{ps}}=\tfrac{1}{2}\right) \\ &\tfrac{1}{2}\begin{cases} l^{\mathrm{ps}} \text{ Up polarity: } j=l^{\mathrm{ps}}+\frac{1}{2} \\ -\left(l^{\mathrm{ps}}+1\right)\ \text{ Dawn polarity: } j=l^{\mathrm{ps}}-\frac{1}{2} \end{cases} \end{aligned} \tag{56.2}$$

The global relativistic energy$E_{\mathrm{nc}}^{s}\left(n,\,\alpha,\,A,\,B,\,H_c,\,H_Y,\,H_H,\,\eta,\,\sigma,\,\chi,\,j,\,l,\,s,\,m\right)$ (E_{nc}^{s}, in short) and $E_{\mathrm{nc}}^{\mathrm{ps}}\left(n,\,\alpha,\,A,\,B,\,H_c,\,H_Y,\,H_H,\,\eta,\,\sigma,\,\chi,\,j,\,l^{\mathrm{ps}},\,s^{\mathrm{ps}},\,m^{\mathrm{ps}}\right)$ ($E_{\mathrm{nc}}^{\mathrm{ps}}$, in short) for the case of spin-1/2 with improved modified Yukawa potential including Coulomb, Yukawa, and Hulthén tensor interactions, in the symmetries of the DDT symmetries, corresponding to the generalized n^{th} excited states:

$$\begin{aligned} &E_{\mathrm{nc}}^{s}=E_{\mathrm{nk}}^{s}+\langle V\rangle_{(\mathrm{nlm})}^{\mathrm{my}}\left(n,\,\alpha,\,A,\,B,\,H_c,\,H_Y,\,H_H\right) \\ &\left[\left(\sigma\aleph+\chi\Omega\right)m+\tfrac{\eta}{2}\begin{cases} l \text{ Up polarity: } j=l+\frac{1}{2} \\ -(l+1)\ \text{ Dawn polarity: } j=l-\frac{1}{2} \end{cases}\right] \end{aligned} \tag{57}$$

and

$$\begin{aligned} &E_{\mathrm{nc}}^{\mathrm{ps}}=E_{\mathrm{nk}}^{\mathrm{ps}}+\langle V^{\mathrm{ps}}\rangle_{(\mathrm{nl}^{\mathrm{ps}}m^{\mathrm{ps}})}^{\mathrm{my}}\left(n,\,\alpha,\,A,\,B,\,H_c,\,H_Y,\,H_H\right) \\ &\left[\left(\sigma\aleph+\chi\Omega\right)m^{\mathrm{ps}}+\tfrac{\eta}{2}\begin{cases} l^{\mathrm{ps}} \text{ Up polarity: } j=l^{\mathrm{ps}}+\frac{1}{2} \\ -\left(l^{\mathrm{ps}}+1\right)\ \text{ Dawn polarity: } j=l^{\mathrm{ps}}-\frac{1}{2} \end{cases}\right] \end{aligned} \tag{58}$$

Where E_{nk}^{s} and $E_{\mathrm{nk}}^{\mathrm{ps}}$ are usual relativistic energies under modified Yukawa potential including a (Coulomb-Yukawa and Hulthén)-like tensor interactions which can be obtained from equations of energy Eqs. (17) and (18).

3.5. STUDY OF IMPORTANT RELATIVISTIC PARTICULAR CASES IN THE CONTEXT OF DDT

In this section, we are about to examine some particular cases regarding the new bound-state energy eigenvalues in Eqs. (57) and (58). We could derive some particular potentials, useful for other physical systems, by adjusting relevant parameters of the *NMYP* model in *DDT* symmetries such as the improved Mie-type potential within three-tensor interactions, the improved Dirac-Coulomb potential within three-tensor interactions, the improved Yukawa potential within the generalized (Yukawa-Coulomb and Hulthén) tensor interactions, the improved inversely quadratic Yukawa potential within the generalized (Yukawa, Coulomb, and Hulthén) tensor interactions.

4.1. DEFORMED DIRAC-THE IMPROVED MIE-TYPE POTENTIAL WITHIN THREE-TENSOR INTERACTIONS

The improved Mie-type potential is obtained from the improved modified Yukawa potential, in DDT symmetries, as follows:

$$V_{mt}\left(r_{nc}\right)=\lim_{\alpha\to 0}V_{my}\left(r_{nc}\right)=\left(\frac{A}{r^2}-\frac{B}{r}+C\right)+\left(\frac{A}{r^4}-\frac{B}{2r^3}\right)L\eta \tag{59}$$

We have used both Eqs. (1), (2), and (34). The first part is Mie-type potential in usual relativistic quantum mechanics[93,94] while the second part $\left(\frac{A}{r^4}-\frac{B}{2r^3}\right)L\eta$ is the effect of topological properties on the Mie-type potential. The global energy for the improved Mie-type under spin sym-

metry and pseudospin symmetry limit with three-tensor interaction is obtained from the equations. (57) and (58) as

$$E_{\rm nc}^{s}=E_{\rm nk}^{s-m}+\langle V\rangle_{(\rm nlm)}^{\rm mt}\left(n,\ A,\ B,\ H_c,\ H_Y,\ H_H\right)$$
$$\left[\left(\sigma\aleph+\chi\Omega\right)m+\frac{\eta}{2}\begin{cases}l\text{ Up polarity: }j=l+\frac{1}{2}\\-(l+1)\text{ Dawn polarity: }j=l-\frac{1}{2}\end{cases}\right] \tag{60}$$

and

$$E_{\rm nc}^{\rm ps}=E_{\rm nk}^{\rm ps-m}+\langle V^{\rm ps}\rangle_{(\rm nl^{ps}m^{ps})}^{\rm mt}\left(n,\ A,\ B,\ H_c,\ H_Y,\ H_H\right)$$
$$\left[\left(\sigma\aleph+\chi\Omega\right)m^{\rm ps}+\frac{\eta}{2}\begin{cases}l^{\rm ps}\text{ Up polarity: }j=l^{\rm ps}+\frac{1}{2}\\-\left(l^{\rm ps}+1\right)\text{ Dawn polarity: }j=l^{\rm ps}-\frac{1}{2}\end{cases}\right] \tag{61}$$

here $E_{\rm nk}^{s-m}$ and $E_{\rm nk}^{\rm ps-m}$ are the energy of Mie-type under spin symmetry and pseudospin symmetry limit with three-tensor interaction which obtained from the energy equations:

$$-\sqrt{\varepsilon_s^2-\beta_sC}=\frac{1}{2}\frac{\beta_sB-2H_YH_H}{n+X\left(A,\ H_c,\ H_Y\right)} \tag{62}$$

$$-\sqrt{\varepsilon_{\rm ps}^2-\beta_{\rm ps}C}=\frac{1}{2}\frac{\beta_{\rm ps}B-2H_YH_H}{n+X\left(A,\ H_c,\ H_Y\right)} \tag{63}$$

with

$$X\left(A,H_c,H_Y\right)=\frac{1}{2}\begin{bmatrix}1-4\beta_sA+4\left(k+H_c\right)\left(k+H_c+1\right)\\+4\left(2k+2H_c+H_Y-1\right)\end{bmatrix}^{1/2}$$

while the new expectation values $\langle V\rangle_{(\rm nlm)}^{\rm mt}$ and $\langle V^{\rm ps}\rangle_{(\rm nl^{ps}m^{ps})}^{\rm mt}$ are determined from Eq. (49) applying the compensation referred to above at the beginning of the current subsection as follows:

$$\begin{pmatrix}\langle V\rangle_{(nlm)}^{mt}\\ \langle V^{ps}\rangle_{(nl^{ps}m^{ps})}^{mt}\end{pmatrix}=\begin{pmatrix}\sum_{\mu=1}^{10}\gamma_{nk}^{\mu s}M_{\mu(nlm)}^{s-mt}\\ \sum_{\mu=1}^{10}\gamma_{nk}^{\mu ps}M_{\mu(nl^{ps}m^{ps})}^{ps-mt}\end{pmatrix} \tag{64}$$

with

$$\left(\gamma_{nk}^{\mu s},Y_{nk}^{\mu ps}\right)=\lim_{\alpha\to0}\left(\beta_{nk}^{\mu s},\beta_{nk}^{\mu ps}\right)$$
$$\text{and }\left(M_{\mu(nlm)}^{s-mt},M_{\mu(nl^{ps}m^{ps})}^{ps-mt}\right)=\lim_{\alpha\to0}\left(M_{\mu(nlm)}^{s-my},M_{\mu(nl^{ps}m^{ps})}^{ps-my}\right)$$

1_DEFORMED DIRAC-THE IMPROVED COULOMB POTENTIAL WITHIN THREE TENSOR INTERACTIONS

When $A=C=0,$ and $\alpha\to0,$ the improved modified Yukawa potential reduces to the improved Coulomb potential. The global energy for this potential under spin symmetry and pseudospin symmetry limit with three-tensor interaction is obtained from equations. (57) and (58) as

$$E_{\rm nc}^{s}=E_{\rm nk}^{s-c}+\langle V\rangle_{(\rm nlm)}^{\rm cp}\left(n,\ B,\ H_c,\ H_Y,\ H_H\right)$$
$$\left[\left(\sigma\aleph+\chi\Omega\right)m+\frac{\eta}{2}\begin{cases}l\text{ Up polarity: }j=l+\frac{1}{2}\\-(l+1)\text{ Dawn polarity: }j=l-\frac{1}{2}\end{cases}\right] \tag{65}$$

and

$$E_{\rm nc}^{\rm ps}=E_{\rm nk}^{\rm ps-c}+\langle V^{\rm ps}\rangle_{(\rm nl^{ps}m^{ps})}^{\rm cp}\left(n,\ B,\ H_c,\ H_Y,\ H_H\right)$$
$$\left[\left(\sigma\aleph+\chi\Omega\right)m^{\rm ps}+\frac{\eta}{2}\begin{cases}l^{\rm ps}\text{ Up polarity: }j=l^{\rm ps}+\frac{1}{2}\\-\left(l^{\rm ps}+1\right)\text{ Dawn polarity: }j=l^{\rm ps}-\frac{1}{2}\end{cases}\right] \tag{66}$$

here $E_{\rm nk}^{s-c}$ and $E_{\rm nk}^{\rm ps-c}$ are the energy of Coulomb potential under spin symmetry and pseudospin symmetry limit with three-tensor interaction which is obtained from the energy equations:

$$-\sqrt{\varepsilon_s^2}=\frac{1}{2}\frac{\beta_sB-2H_YH_H}{n+Y\left(A,\ H_c,\ H_Y\right)} \tag{67}$$

$$-\sqrt{\varepsilon_{\rm ps}^2}=\frac{1}{2}\frac{\beta_{\rm ps}B-2H_YH_H}{n+Y\left(A,\ H_c,\ H_Y\right)} \tag{68}$$

With

while the new expectation values $\langle V\rangle_{(\rm nlm)}^{\rm cp}$ and $\langle V^{\rm ps}\rangle_{(\rm nl^{ps}m^{ps})}^{\rm cp}$ are determined from Eq. (49) applying the compensation referred to above at the beginning of the current subsection as follows:

$$\sum_{\mu=1}^{10}\delta_{\rm nk}^{\mu\rm ps}M_{\mu(\rm nl^{ps}m^{ps})}^{\rm ps-cp} \tag{69}$$

with

$$\left(\delta_{nk}^{\mu s},\delta_{nk}^{\mu ps}\right)=\lim_{(\alpha,A,C)\to(0,0,0)}\left(\beta_{nk}^{\mu s},\beta_{nk}^{\mu ps}\right)$$
$$\text{and }\left(M_{\mu(nlm)}^{s-cp},M_{\mu(nl^{ps}m^{ps})}^{ps-cp}\right)=\lim_{(\alpha,A,C)\to(0,0,0)}\left(M_{\mu(nlm)}^{s-my},M_{\mu(nl^{ps}m^{ps})}^{ps-my}\right)$$

2_DEFORMED DIRAC-THE IMPROVED YUKAWA POTENTIAL WITHIN THREE TENSOR INTERACTIONS

When $A=C=0$the improved modified Yukawa potential is reduced to the improved Yukawa potential. The global energy for this potential under spin symmetry and pseudospin symmetry limit with three-tensor interaction is obtained from equations. (57) and (58) as

$$E_{\rm nc}^{s}=E_{\rm nk}^{s-y}+\langle V\rangle_{(\rm nlm)}^{\rm yp}\left(n,\ \alpha,\ B,\ H_c,\ H_Y,\ H_H\right)$$
$$\left[\left(\sigma\aleph+\chi\Omega\right)m+\frac{\eta}{2}\begin{cases}l\text{ Up polarity: }j=l+\frac{1}{2}\\-(l+1)\text{ Dawn polarity: }j=l-\frac{1}{2}\end{cases}\right] \tag{70}$$

and

$$E_{\rm nc}^{\rm ps}=E_{\rm nk}^{\rm ps-y}+\langle V^{\rm ps}\rangle_{(\rm nl^{ps}m^{ps})}^{\rm cp}\left(n,\ \alpha,\ B,\ H_c,\ H_Y,\ H_H\right)$$
$$\left[\left(\sigma\aleph+\chi\Omega\right)m^{\rm ps}+\frac{\eta}{2}\begin{cases}l^{\rm ps}\text{ Up polarity: }j=l^{\rm ps}+\frac{1}{2}\\-\left(l^{\rm ps}+1\right)\text{ Dawn polarity: }j=l^{\rm ps}-\frac{1}{2}\end{cases}\right] \tag{71}$$

here $E_{\rm nk}^{s-y}$ and $E_{\rm nk}^{\rm ps-y}$ are the energy of Yukawa potential under spin symmetry and pseudospin symmetry limit with three-tensor interaction which are obtained from the energy equations:

$$-\sqrt{\frac{\varepsilon_s^2}{\alpha^2}-\frac{\beta_sC}{\alpha^2}+\left(k+H_c\right)\ \left(k+H_c+1\right)}=\frac{1}{2}\{n+\sigma_s$$
$$\left.+\frac{\frac{\beta_sB}{\alpha}+\left(k+H_c\right)\left(k+H_c+1\right)}{n+\sigma_s}-\frac{H_Y\left(H_Y-1\right)+\frac{H_Y}{\alpha}\left(\frac{H_Y}{\alpha}+2H_Y\right)}{n+\sigma_s}\right\} \tag{72}$$

$$-\sqrt{\frac{\varepsilon_{ps}^2}{\alpha^2}-\frac{\beta_{ps}C}{\alpha^2}+\left(k+H_c\right)\left(k+H_c-1\right)}=\frac{1}{2}\{n+\sigma_{ps}$$
$$\left.+\frac{\frac{\beta_{ps}B}{\alpha}+\left(k+H_c\right)\left(k+H_c-1\right)}{n+\sigma_{ps}}-\frac{H_Y\left(H_Y+1\right)+\frac{H_Y}{\alpha}\left(\frac{H_Y}{\alpha}+2H_Y\right)}{n+\sigma_{ps}}\right\} \tag{73}$$

With

$$\sigma_s=\frac{1}{2}+\frac{1}{2}\{1+4\left(k+H_c\right)\ \left(k+H_c-1\right)+4\left(2k+2H_c+H_Y-1\right)H_Y$$
$$\left.+4\frac{H_H}{\alpha}\left(\frac{H_Y}{\alpha}+2k+2H_c+2H_Y-1\right)\right\}^{\frac{1}{2}} \tag{74}$$

$$\sigma_{\rm ps}=\frac{1}{2}+\frac{1}{2}\{1+4\left(k+H_c\right)\ \left(k+H_c+1\right)+4\left(2k+2H_c+H_Y-1\right)H_Y$$
$$\left.+4\frac{H_H}{\alpha}\left(\frac{H_Y}{\alpha}+2k+2H_c+2H_Y-1\right)\right\}^{\frac{1}{2}} \tag{75}$$

while the new expectation values $\langle V\rangle_{(\rm nlm)}^{\rm yp}$ and $\langle V^{\rm ps}\rangle_{(\rm nl^{ps}m^{ps})}^{\rm yp}$ are determined from Eq. (49) applying the compensation referred to above at the beginning of the current subsection as follows:

$$\begin{cases}\langle V\rangle_{(nlm)}^{yp}=\sum_{\mu=1}^{10}\varepsilon_{nk}^{\mu s}M_{\mu(nlm)}^{s-yp}\\ \langle V^{ps}\rangle_{(nl^{ps}m^{ps})}^{yp}=\sum_{\mu=1}^{10}\varepsilon_{nk}^{\mu ps}M_{\mu(nl^{ps}m^{ps})}^{ps-yp}\end{cases} \tag{76.1}$$

with

$$\begin{cases}\left(\varepsilon_{nk}^{\mu s},\varepsilon_{nk}^{\mu ps}\right)=\lim_{(A,C)\to(0,0)}\left(\beta_{nk}^{\mu s},\beta_{nk}^{\mu ps}\right)\\ \left(M_{\mu(nlm)}^{s-yp},M_{\mu(nl^{ps}m^{ps})}^{ps-yp}\right)=\lim_{(A,C)\to(0,0)}\left(M_{\mu(nlm)}^{s-my},M_{\mu(nl^{ps}m^{ps})}^{p-my}\right)\end{cases} \tag{76.2}$$

3_DEFORMED DIRAC-THE IMPROVED INVERSELY QUADRATIC YUKAWA POTENTIAL WITHIN THREE TENSOR INTERACTIONS

When $B=C=0$ and $A=-V_0$, the improved modified Yukawa potential reduces to the improved inversely quadratic Yukawa potential. The global energy for this potential under spin symmetry and pseudospin symmetry limit with three-tensor interaction is obtained from equations. (57) and (58) as

$$E_{\rm nc}^{s-{\rm iq}}=E_{\rm nk}^{s-{\rm iq}}+\langle V\rangle_{({\rm nlm})}^{\rm iq}\left(n,\ \alpha,\ V_0,\ H_c,\ H_Y,\ H_H\right)$$
$$\left[\left(\sigma\aleph+\chi\Omega\right)m+\frac{\eta}{2}\begin{cases}l\\ \text{Up polarity: } j=l+\frac{1}{2}\\ -(l+1)\\ \text{Dawn polarity: } j=l-\frac{1}{2}\end{cases}\right] \tag{77}$$

and

$$E_{\rm nc}^{{\rm ps}-{\rm iq}}=E_{\rm nk}^{{\rm ps}-{\rm iq}}+\langle V^{\rm ps}\rangle_{({\rm nl}^{\rm ps}m^{\rm ps})}^{\rm iq}\left(n,\ \alpha,\ V_0,\ H_c,\ H_Y,\ H_H\right)$$
$$\left[\left(\sigma\aleph+\chi\Omega\right)m^{\rm ps}+\frac{\eta}{2}\begin{cases}l^{\rm ps}\\ \text{Up polarity: } j=l^{\rm ps}+\frac{1}{2}\\ -\left(l^{\rm ps}+1\right)\\ \text{Dawn polarity: } j=l^{\rm ps}-\frac{1}{2}\end{cases}\right] \tag{78}$$

here $E_{\rm nk}^{s-{\rm iq}}$ and $E_{\rm nk}^{{\rm ps}-{\rm iq}}$ are the energy of inversely quadratic Yukawa potential under spin symmetry and pseudospin symmetry limit with three-tensor interaction which are obtained from the energy equations:

$$-\sqrt{\frac{\varepsilon_s^2}{\alpha^2}-\frac{\beta_sC}{\alpha^2}+\left(k+H_c\right)\ \left(k+H_c+1\right)}=\frac{1}{2}(n+\sigma_s$$
$$+\frac{\frac{-\beta_sV_0}{\alpha}+\left(k+H_c\right)\left(k+H_c+1\right)}{n+\sigma_s}$$
$$-\frac{H_Y(H_Y-1)+\frac{H_Y}{\alpha}\left(\frac{H_Y}{\alpha}+2H_Y\right)}{n+\sigma_s}) \tag{79}$$

$$-\sqrt{\frac{\varepsilon_{ps}^2}{\alpha^2}-\frac{\beta_{ps}C}{\alpha^2}+\left(k+H_c\right)\ \left(k+H_c-1\right)}=\frac{1}{2}(n+\sigma_{\rm ps}$$
$$+\frac{\frac{-\beta_{ps}V_0}{\alpha}+\left(k+H_c\right)\left(k+H_c-1\right)}{n+\sigma_{\rm ps}}$$
$$-\frac{H_Y(H_Y+1)+\frac{H_Y}{\alpha}\left(\frac{H_Y}{\alpha}+2H_Y\right)}{n+\sigma_{\rm ps}}) \tag{80}$$

with

$$\sigma_s=\frac{1}{2}\{1+(1+4\beta_sV_0+4\left(k+H_c\right)\ \left(k+H_c-1\right)$$
$$+4\left(2k+2H_c+H_Y-1\right)H_Y$$
$$+4\frac{H_H}{\alpha}\left(\frac{H_Y}{\alpha}+2k+2H_c+2H_Y-1\right)\Big\}^{\frac{1}{2}} \tag{81}$$

$$\sigma_{ps}=\frac{1}{2}(1+(1+4\beta_{ps}V_0+4\left(k+H_c\right)\left(k+H_c+1\right)$$
$$+4\left(2k+2H_c+H_Y-1\right)H_Y$$
$$+4\frac{H_H}{\alpha}\left(\frac{H_Y}{\alpha}+2k+2H_c+2H_Y-1\right)\Big\}^{1/2} \tag{82}$$

while the new expectation values $\langle V\rangle_{({\rm nlm})}^{\rm iq}$ and $\langle V^{\rm ps}\rangle_{({\rm nl}^{\rm ps}m^{\rm ps})}^{\rm iq}$ are determined from Eq. (49) applying the compensation referred to above at the beginning of the current subsection as follows:

$$\sum_{\mu=1}^{10}\varepsilon_{\rm nk}^{\mu{\rm ps}}M_{\mu({\rm nl}^{\rm ps}m^{\rm ps})}^{{\rm ps}-{\rm yp}} \tag{83}$$

with

$$\left(\varepsilon_{nk}^{\mu s},\varepsilon_{nk}^{\mu ps}\right)=\lim_{(B,C)\to(0,0)}\left(\beta_{nk}^{\mu s},\beta_{nk}^{\mu ps}\right)\ \text{and}$$
$$\left(M_{\mu(nlm)}^{s-iq},M_{\mu(nl^{ps}m^{ps})}^{ps-iq}\right)=\lim_{(B,C)\to(0,0)}\left(M_{\mu(nlm)}^{s-my},M_{\mu(nl^{ps}m^{ps})}^{ps-my}\right)$$

4. DEFORMED SCHRÖDINGER NMYP PROBLEMS IN NREQM SYMMETRIES

To realize a study of the nonrelativistic limit, in extended nonrelativistic quantum mechanics NREQM symmetries of the NMYP, two steps must be applied, the first step corresponds to the nonrelativistic limit, in usual nonrelativistic quantum energy. This is done by applying the following steps, we replace $\left(H_c,\ H_Y,\ H_H,\ C_s\right)$, $E_{\rm nk}^s+M$, $E_{\rm nk}^s-M$, $k(k+1)$, $F_{\rm nk}(r)$ and ε_s^2 by $(0,\ 0,\ 0,\ 0)$, 2μ, $E_{\rm nl}^{\rm nr}$, $l(l+1)$, $R_{\rm nk}(r)$ and $-2\mu E_{\rm nl}^{\rm nr}$, respectively, allows us to obtain the nonrelativistic energy levels as:

$$E_{\rm nl}^{\rm nr}=\frac{\alpha^2}{2\mu}l(l+1)-\frac{2\mu C}{\alpha^2}-\frac{\alpha^2}{8\mu}$$
$$\left[\frac{2\mu\left(A+\frac{B}{\alpha}\right)+l(l+1)}{n+\sigma_{\rm nr}}+n+\sigma_{\rm nr}\right]^2 \tag{84}$$

with

$$\sigma_{nr}=\frac{1}{2}(1+\sqrt{1-8\mu A+4l(l+1)})$$

Now, the second step corresponds to the new coefficients $Q_{\rm nl}^1=\alpha^4l(l+1)$, $Q_{\rm nl}^2=-\mu B\alpha^3$, $Q_{\rm nl}^3=0$, $Q_{\rm nl}^{4s}=-\mu B\alpha^3$, $Q_{\rm nl}^5=2\mu A\alpha^4$ and $Q_{\rm nl}^6=2\mu A\alpha^4$ which were obtained by applying the previous limits. Allows us to reexport the relativistic expectation values $\langle V\rangle_{({\rm nlm})}^{\rm my}\left(n,\ \alpha,\ A,\ B,\ H_c,\ H_Y,\ H_H\right)$ of spin symmetry in Eq. (49) from the corresponding nonrelativistic expectation values $\langle V\rangle_{({\rm nlm})}^{{\rm nr}-{\rm my}}(n,\ \alpha,\ A,\ B)$ as:

$$\sum_{\mu=1}^{6}Q_{\rm nl}^{\mu}M_{\mu({\rm nlm})}^{s-{\rm my}} \tag{85}$$

This permuted expressing the nonrelativistic correction energy $\Delta E_{{\rm nc}-{\rm nr}}^{\rm my}\left(n,\ \alpha,\ A,\ B,\ \eta,\ \sigma,\ \chi,\ j,\ l,\ s,\ m\right)\ \equiv\Delta E_{{\rm nc}-{\rm nr}}^{\rm my}$ produced by the new Yukawa potential problems as

$$\Delta E_{{\rm nc}-{\rm nr}}^{\rm my}=\langle V\rangle_{({\rm nlm})}^{{\rm nr}-{\rm my}}[\left(\sigma\aleph+\chi\Omega\right)m|] \tag{86}$$

The global NR energy $E_{{\rm nc}-{\rm nr}}^{\rm my}(n,\ \alpha,\ A,\ B,\ \eta,\ \sigma,\ \chi,\ j,\ l,\ s,\ m)\equiv\Delta E_{{\rm nc}-{\rm nr}}^{\rm my}$ produced with the improved modified Yukawa potential in ENRQM symmetries as a result of the topological properties of deformation space-space is the sum of usual energy $E_{\rm nl}^{\rm my}$ in Eq. (84) under modified Yukawa potential in NRQM symmetries and the obtained correction $\Delta E_{{\rm nc}-{\rm nr}}^{\rm my}$ in Eq. (86) as follows:

$$E_{{\rm nc}-{\rm nr}}^{\rm my}=\frac{\alpha^2}{2\mu}l(l+1)-\frac{2\mu C}{\alpha^2}$$
$$-\frac{\alpha^2}{8\mu}\left[\frac{2\mu\left(A+\frac{B}{\alpha}\right)+l(l+1)}{n+\sigma_{\rm nr}}+n+\sigma_{\rm nr}\right]^2[\left(\sigma\aleph+\chi\Omega\right)m|]$$
$$+\langle V\rangle_{({\rm nlm})}^{{\rm nr}-{\rm my}} \tag{87}$$

It should be noted that the corrected energy ΔE_{nc-nr}^{my} expressed in Eq. (86) is due to the effect of the perturbed potential $V_{nr-pert}^{my}(r)$:

$$V_{nr-pert^{my}}(r)=l(l+1)\,r^{-4}\mathbf{L}\eta-\frac{\partial V_{\rm my}(r)}{\partial r}\frac{\mathbf{L}\eta}{2r}+O\left(\eta^2\right) \tag{88}$$

The first term in Eq. (80) is due to the centrifuge term $l(l+1)\,r_{\rm nc}^{-2}$ in ENRQM symmetries which equals the usual centrifuge term $l(l+1)\,r^{-2}$ plus the perturbative centrifuge term $l(l+1)\,r^{-4}\mathbf{L}\eta$ while the second term is produced with the effect of the NMYP. This is one of the most important new results of this research.

5. SPIN-AVERAGED MODIFIED MASS SPECTRA OF HLM UNDER NMYP

In this section, we use the non-relativistic energies that represent the binding energy between the quark and the antiquark to determine the modified spin-averaged mass spectra of heavy and heavy-light mesons *HLM* such as $c\bar{c}$,$b\bar{b}$, $b\bar{c}$, $b\bar{s}$, $c\bar{s}$ and $b\bar{q}$, q= (u, d) under NMYP by using the following formula:

$$M_{\rm nl}^{\rm myp}=m_Q+m_{\bar{q}}+E_{\rm nl}^{\rm nr}\to M_{\rm nl}^{\rm mod}=m_Q+m_{\bar{q}}$$
$$+\begin{cases}\frac{1}{3}\left(E_{\rm nl}^{{\rm nc}-u}+E_{\rm nl}^{{\rm nc}-m}+E_{\rm nl}^{{\rm nc}-l}\right)\ \text{for spin-1}\\ E_{\rm nl}^{\rm nc}\ \text{for spin-0}\end{cases} \tag{89}$$

The LHS of Eq. (89) describes spin-averaged mass spectra of *HLM* in usual quantum mechanics symmetries,[95,96] while

the RHS is our generalization to this equation in ENRQM symmetries, m_Q and $m_{\bar{q}}$ are the quark mass and the antiquark mass, $M_{\rm nl}^{\rm myp}$ is the spin-averaged mass spectra of *HLM* such as $c\bar{c}$,$b\bar{b}$, $b\bar{c}$, $b\bar{s}$, $c\bar{s}$ and $b\bar{q}$, q= (u, d) under *MYP* in usual NRQM symmetries, $E_{\rm nl}^{\rm nr}$ is the nonrelativistic energy under MYP which determined in Eq. (84) while ($E_{\rm nl}^{{\rm nc}-u}$, $E_{\rm nl}^{{\rm nc}-m}$, $E_{\rm nl}^{{\rm nc}-l}$) are the modified energies of *HLM* which have spin-1 while $E_{\rm nl}^{\rm nc}$ is the modified energies of *HLM* which have spin-0. We need to replace the factor $F(j,\ l,\ s)$ with new generalized values $F^n(j,\ l,\ s)$ as follows:

$$F^n(j,l,s)=\frac{[j(j+1)-l(l+1)-s(s+1)]}{2}$$
$$\begin{cases} \frac{l}{2}\ \text{For:}\ j=l+1\ \text{and}\ s=1 \\ -1\ \text{For:}\ j=l\ \text{and}\ s=1 \\ -(l+1)\ \ \text{For:}\ j=l-1\ \text{and}\ s=1 \\ 0\ \text{For:}\ j=l\ \text{and}\ s=0 \end{cases} \tag{90}$$

Allows us to obtain ($E_{\rm nl}^{{\rm nc}-u}$, $E_{\rm nl}^{{\rm nc}-m}$, $E_{\rm nl}^{{\rm nc}-l}$) and $E_{\rm nl}^{\rm nc}$ of the *HLM* such as $c\bar{c}$, $b\bar{b}$, $b\bar{c}$, $b\bar{s}$, $c\bar{s}$, and $b\bar{q}$, q = (u, d) as:

$$\begin{cases} E_{\rm nl}^{{\rm nc}-u}=E_{\rm nl}^{\rm nr}+\langle V\rangle_{\rm (nlm)}^{\rm nr-my}\left[(\sigma\aleph+\chi\Omega)m+\eta\frac{l}{2}\right] \\ \text{For:}\ j=l+1\ \text{and}\ s=1 \\ E_{\rm nl}^{{\rm nc}-m}=E_{\rm nl}^{\rm nr}+\langle V\rangle_{\rm (nlm)}^{\rm nr-my}\left[(\sigma\aleph+\chi\Omega)m-\eta\right] \\ \text{For:}\ j=l\ \text{and}\ s=1 \\ E_{\rm nl}^{{\rm nc}-l}=E_{\rm nl}^{\rm nr}+\langle V\rangle_{\rm (nlm)}^{\rm nr-my}\left[(\sigma\aleph+\chi\Omega)m-\eta(l+1)\right] \\ \text{For:}\ j=l-1\ \text{and}\ s=1 \\ E_{\rm nl}^{\rm nc}=E_{\rm nl}^{\rm nr}+\langle V\rangle_{\rm (nlm)}^{\rm nr-my}\left[(\sigma\aleph+\chi\Omega)m\right] \\ \text{For:}\ j=l\ \text{and}\ s=0 \end{cases} \tag{91}$$

By substituting Eqs. (91) and (90) into Eq. (89), the new mass spectrum of the meson systems in ENRQM symmetries under the *NMYP* for any arbitrary radial and angular momentum quantum numbers become:

$$M_{\rm nl}^{\rm mod}=M_{\rm nl}^{\rm myp}+\langle V\rangle_{\rm (nlm)}^{\rm nr-my}$$
$$\begin{cases} \left[(\sigma\aleph+\chi\Omega)\ m-\frac{\eta}{3}\right]\ \text{for}\ s=1 \\ \left[(\sigma\aleph+\chi\Omega)\ m\right]\ \text{for}\ s=0 \end{cases} \tag{92}$$

Thus the spin-averaged mass spectra $M_{\rm nl}^{\rm myp}$ of *HLM* such as $c\bar{c}$,$b\bar{b}$, $b\bar{c}$, $b\bar{s}$, $c\bar{s}$ and $b\bar{q}$, q= (u, d) under the MYP in usual NRQM symmetries:

$$M_{nl}^{myp}=m_Q+m_{\bar{q}}$$
$$-\frac{4\alpha^2}{2\mu}\left[\frac{D(n,l,\mu,V_0,A,B)}{(2n+1)+2\sqrt{\frac{1}{4}+l(l+1)-2V_0B^2\mu}}\right]^2-V_0A^2 \tag{93}$$

is extended to include $\delta M_{\rm nl}^{\rm mod}$ in ENRQM symmetries:

$$\delta M_{\rm nl}^{\rm mod}=\delta M_{\rm nl}^{\rm mod}-M_{\rm nl}^{\rm myp}=\langle Z\rangle_{\rm (nlm)}^{\rm qy-nr}$$
$$\begin{cases} \left[(\sigma\aleph+\chi\Omega)\ m-\frac{\eta}{3}\right]\ \text{for}\ s=1 \\ \left[(\sigma\aleph+\chi\Omega)\ m\right]\ \text{for}\ s=0 \end{cases} \tag{94}$$

This allows the realized physical limit $\lim\limits_{(\eta,\ \sigma,\ \chi)\to(0,\ 0,\ 0)} M_{\rm nl}^{\rm mod}=M_{\rm nl}^{\rm myp}$ to be achieved. It is worth mentioning that for the three- simultaneous limits $(\eta,\ \sigma,\ \chi)\to(0,\ 0,\ 0)$, we recover the energy equations for the spin symmetry and the p-spin symmetry, under the modified Yukawa potential within the generalized (Coulomb, Yukawa, and Hulthén)-like tensor interactions which are treated in the third Ref.

6. SUMMARY AND CONCLUSIONS

In summary, this work presents an approximate analytical solution of the 3-dimensional deformed Dirac equation with a new modified Yukawa potential within three tensor interactions (Coulomb-like, Yukawa-like, and Hulthén-type potentials) under pseudospin and spin symmetry limits with arbitrary spin-orbit coupling quantum numbers k . To do so we have dealt with the centrifugal potential term using the Greene-Aldrich approximation. In this manner, we have obtained the new approximate bound-state energies that appeared sensitive to the quantum numbers (j, k, l, $l^{\rm ps}$, s, $s^{\rm ps}$, m, $m^{\rm ps}$), the potential depths (A, B, H_c, H_Y, H_H) of the studied potentials, the range of the potentials α , and noncommutativity parameters (η, σ, χ) under the condition of spin and pseudospin symmetry. As we know, the new modified Yukawa potential within three tensor interactions (Coulomb-like, Yukawa-like, and Hulthén-type potentials) reduces to the improved Mie-type potential within three tensor interactions, the improved Dirac-Coulomb potential within three tensor interactions, the improved Yukawa potential within the generalized (Yukawa-Coulomb and Hulthén) tensor interactions, and the improved inversely quadratic Yukawa potential within the generalized (Yukawa, Coulomb, and Hulthén) tensor interactions. We also ended our research with this treatment of the nonrelativistic limit of the NMYP three tensor interactions (Coulomb-like, Yukawa-like, and Hulthén-type potentials) in ENRQM symmetries. It is worth mentioning that, for all cases, to make the three simultaneous limits $(\eta,\ \sigma,\ \chi)\to(0,\ 0,\ 0)$, the ordinary physical quantities are recovered in ref.[3] Finally, a feature of a noncommutative geometry on the 3-dimensional deformed Dirac equation with a new modified Yukawa potential within three tensor interactions (Coulomb-like, Yukawa-like, and Hulthén-type potentials) would be the presence of many physics phonemes which usually appear automatically such as spin-orbit and pseudospin-orbit, modified Zeeman effect and others and cause the behavior of topological properties of deformed space-space.

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