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RESEARCH ARTICLE

# Generation Transcritical Flow Influenced by Dissipation over a Hole

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## ABSTRACT

Transcritical flow of a stratified fluid over an obstacle for negative forcing amplitude (hole) that generation upstream and downstream, connected by an unsteady solution is examined. In the weakly nonlinear, weakly dispersive regime, the problem is formulated in the forced Korteweg-de Vries–Burgers framework. This is done by including the influence of the viscosity of the fluid beyond the Korteweg–de Vries approximation. The results show that the influence of viscosity is crucial in determining various wave properties, including the amplitudes of solitary waves in the upstream and downstream directions, as well as the widths of the bores. We focused here on weak damping and the outcomes are prepared for transcritical, supercritical, and subcritical flows. In general, the outcomes are not qualitatively similar to those from the forced Korteweg-de–Vries equation when the value of the viscous is small, interesting differences emerge as the magnitude of the value of viscous increases.

**Keywords:** Korteweg-de Vries equation, Soliton, Resonant flow, Viscous flow, Modulation theory, Dispersive shock waves, Undular bores

## 1. Introduction

Waves are a significant occurrence in nature, playing a crucial role in various important physical phenomena [1–11]. One of the most common and important types of waves are water waves. The study of water waves is important in both engineering and geophysical contexts, such as understanding flows in estuaries and deep ocean overflows. One specific area that has been extensively studied in the last 40 years is the transcritical (resonant) flow of a stratified fluid over an obstacle or through a contraction over topography. Many theoretical and numerical solutions have been published for the resonant flow over topography in the weakly nonlinear and long wave regime for the case when the forcing has positive polarity, see Albalwi [12, 13], for comprehensive reviews of these studies.

Several authors have investigated this phenomenon for negative forcing amplitude (hole). For example, Grimshaw & Smyth presented numerical solutions that showed that no steady state forms upstream of the topography [14]. Zhang and Chwang conducted a numerical study of the full Euler equations for transcritical flow over a step. Their findings revealed that a positive topography, or bump, causes upstream advancing solitary waves due to the forward step forcing. Conversely, a negative topography, or hollow, results in downstream-radiating waves induced by the backward step forcing. Furthermore, they observed that the characteristics of the upstream and downstream undular bores closely align with the principles of modulation theory for the forced Korteweg-de Vries equation [15]. This work was developed by Grimshaw et al. [16]. They investigated the flow over a hole numerically by using the forced Korteweg-de Vries equation, consisting of a step down followed by a step up, the two generated wavetrains interact over the hole. Ee et al. studied transcritical flow of a stratified fluid over a hole described by the forced Korteweg-de Vries equation [17]. Recently, the effect of including an additional cubic nonlinear

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term in the forced Korteweg-de Vries equation (the forced Gardner equation) has been investigated in some special circumstances [18]. In the work of Grimshaw & Maleewong, upstream and downstream undular bores were generated and found to interact in the case of a long hole (comprising a negative obstacle followed by a positive obstacle) [19]. The mechanism for the upstream generation of solitary waves using the forced Gardner equation is studied [20].

The primary purpose of this present paper to investigate the effect of viscosity on the flow of a stratified fluid over a hole. In particular, the resonant flow over a hole will be considered in the weakly nonlinear, long wave limit, and dissipation effects. The viscous term will be added to the Korteweg-de Vries approximation so that the flow is governed by a forced Korteweg-de Vries-Burgers' equation. The paper is organized as follows. In Section 2 outlines of the forced Korteweg-de Vries-Burgers' equation. In Section 3 comparisons are made between the numerical simulations of viscous and inviscid flow results. In Section 4 conclusions are given.

## 2. The forced Korteweg-de Vries-Burgers' equation

The equation governing the transcritical flow of a fluid over topography is the forced Korteweg-de Vries-Burgers' equation, which is non-dimensional and normalized

$$-\frac{\partial u}{\partial t}-\Delta\frac{\partial u}{\partial x}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}+\gamma\frac{\partial^2 u}{\partial x^2}+\frac{dG(x)}{dx}=0, \tag{1}$$

where $x$ represents the space variable and $t$ represents the time variable. Here, the amplitude of the relevant linear long-wave model is $u(x,t)$. The detuning parameter $\Delta$ measures the deviation from the exact

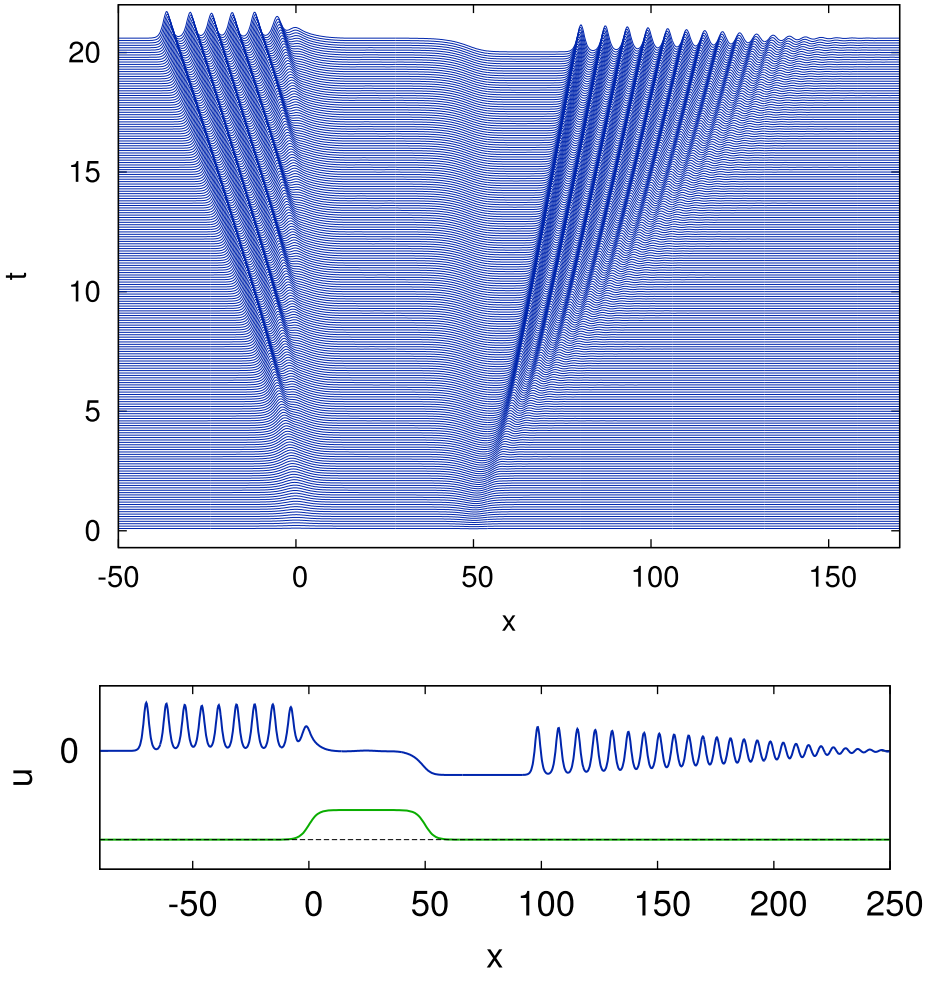


**Fig. 1.** A perspective view of the forced Korteweg-de Vries Eq. (1) is shown in the $x-t$ plane with time up to $t=20$ (top) and the surface profile $u$ and bathymetry $G$ versus $x$ at $t=20$ (bottom). The forcing function is (8) and $g_0=1$. The other parameters are $\Delta=0$ and $W=0.25$.

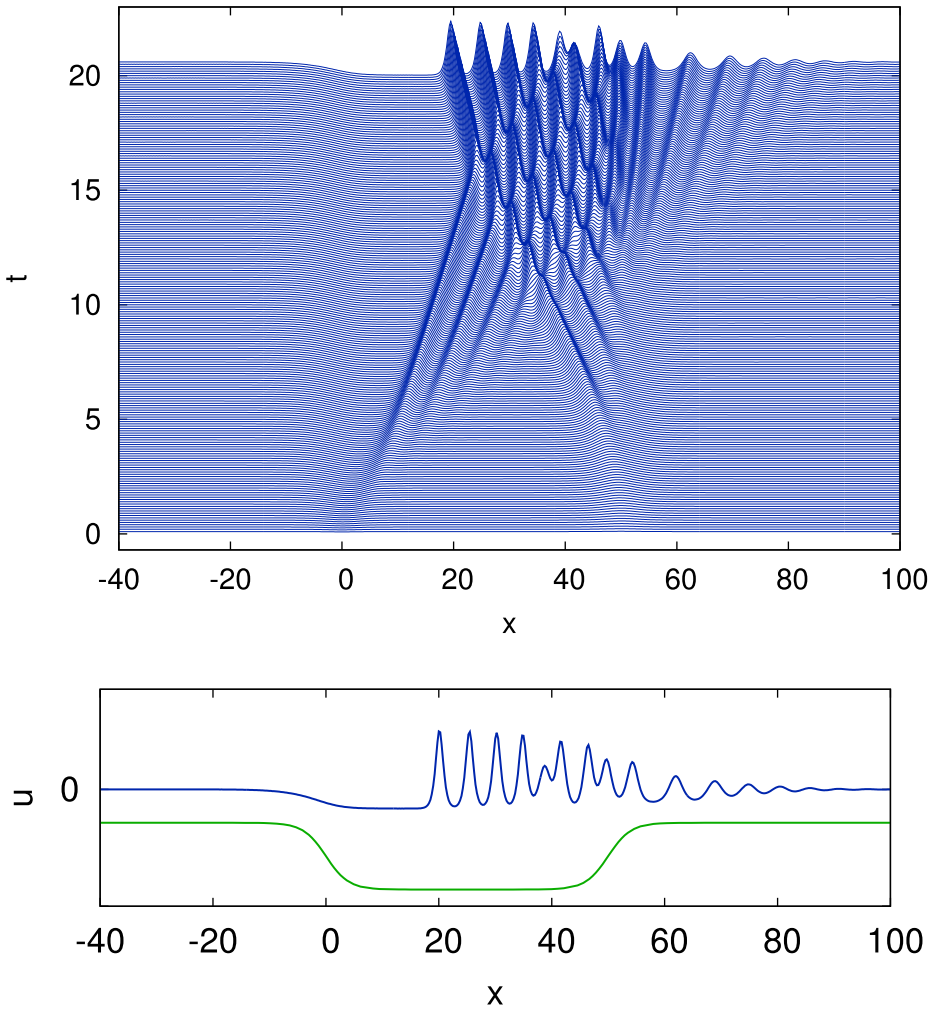


**Fig. 2.** Solution of the forced Korteweg-de Vries Eq. (1) for the same situation as Fig. 1 with $g_0 = -1$.

resonance case of $\Delta = 0$, while $\gamma$ is the coefficient of the damping term. $G(x)$ is the obstacle profile. This equation is an effective model for simulating and explaining the amplitudes of solitary waves both upstream and downstream, as well as the width of the bore [21, 22].

In this study, the flow is assumed to begin from a state of rest. Therefore,

$$u(x, 0) = 0. \tag{2}$$

The Eq. (1) can be solved asymptotically using the method of characteristics

$$\frac{du}{dt} = \frac{dG(x)}{dx} \quad \text{on} \quad \frac{dx}{dt} = \Delta - 6u. \tag{3}$$

This system can easily be integrated explicitly

$$A(u) = G(x) - G(x_c) \qquad A(u) = \Delta u - 3u^2. \tag{4}$$

The position $x_c$ is taken to lie sufficient distance from the obstacle, so $-\infty < x < x_c$ and $u = 0$ with derivatives zero. In the hydraulic limit, a shock is formed when characteristics intersect, and the undular bore solutions of the full equation must be used to determine them. Hence, the shock speed $V$ can be easily determined by using the following equation

$$V[u] = [A(u)]^{+}_{-}. \tag{5}$$

Here, $V \to \pm\infty$ as $x \to \pm\infty$. There are two main scenarios. In the first, the characteristics intersect at the forcing point, causing the formation of undular bores that propagate either upstream or downstream. In the second, the characteristics move away from the forcing region before possibly intersecting and creating an undular bore.

The nonlinear hyperbolic Eq. (1) has two stable solutions. In this scenario, the preferred solution for the steady flow is one of these two stable

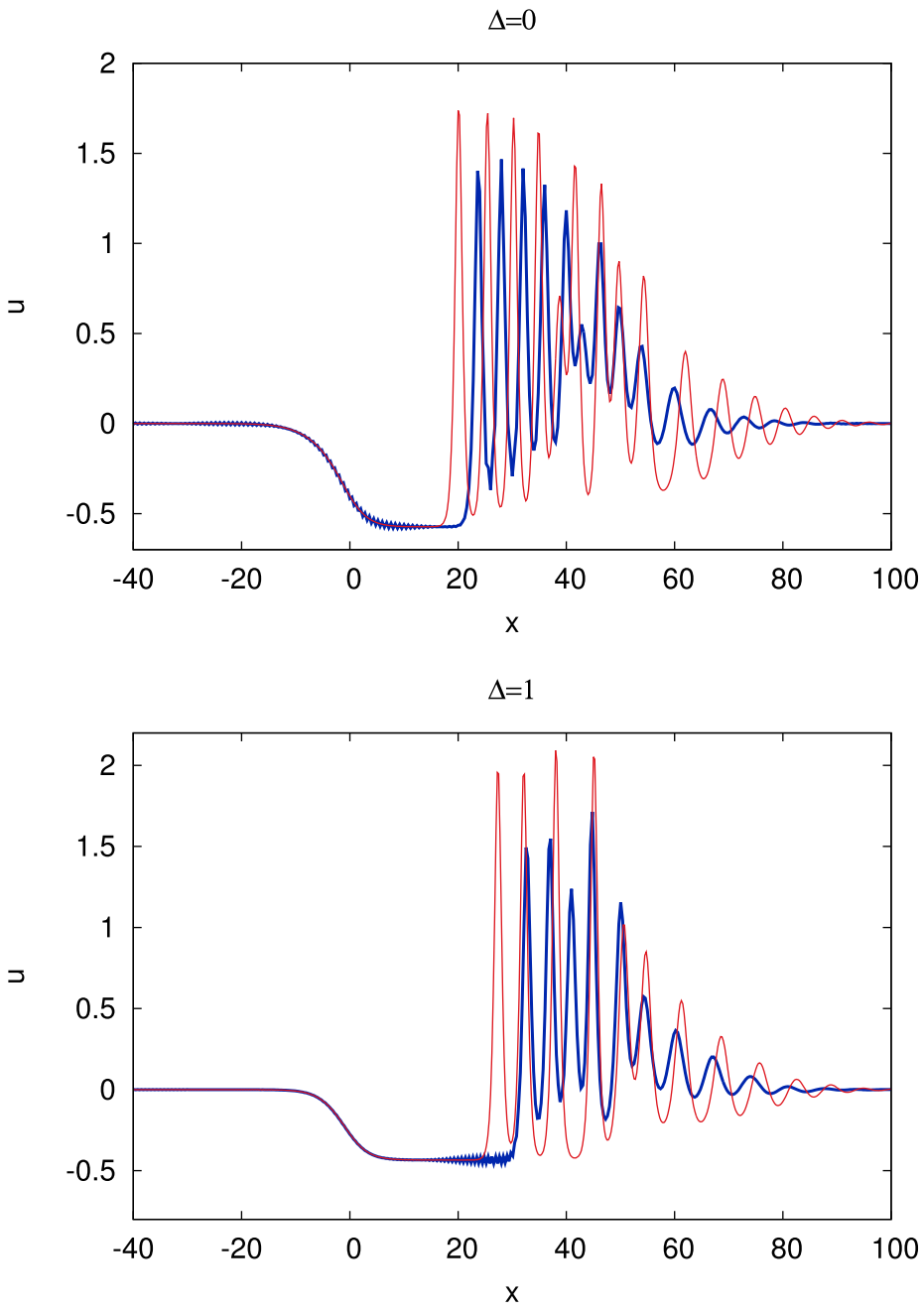


**Fig. 3.** The surface profile $u$ versus $x$ at $t = 20$. Displayed are computational results of the forced Korteweg-de Vries-Burgers' Eq. (1) for $\gamma = 0.05$ (blue solid line) and the forced Korteweg-de Vries equation (red line). The forcing is (8) with $g_0 = -1$ and $W = 0.25$.

solutions

$$u_s = \begin{cases} \frac{1}{6}\left(1 - \gamma\frac{1}{6}\right)[\Delta + \sqrt{12(g_0 - G(x))}], & x < 0, \\ \frac{1}{6}\left(1 - \gamma\frac{1}{6}\right)[\Delta - \sqrt{12(g_0 - G(x))}], & x > 0. \end{cases} \tag{6}$$

Here $g_0$ is a parameter measuring the height of the topography. The flow resonance will occur when $\Delta$ lies in the regime

$$-\sqrt{12g_0} < \Delta < \sqrt{12g_0}. \tag{7}$$

Note that the width of the transcritical band, $2\sqrt{12g_0}$, is the same as the one provided by the forced Korteweg-de Vries equation, without any alterations. In this work, we will focus on the situation in which the forcing exhibits a negative polarity, indicating a

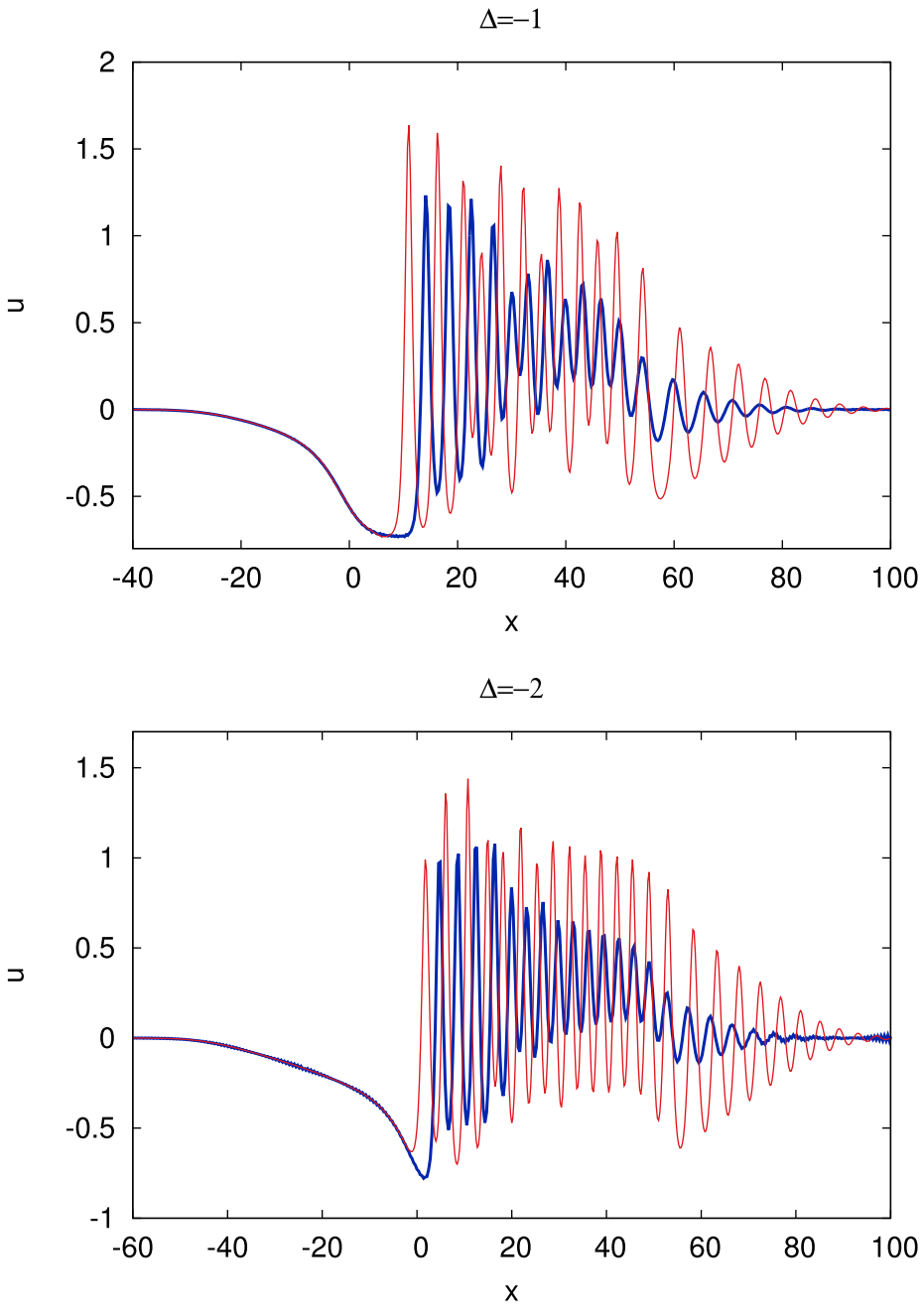


**Fig. 4.** Subcritical solutions u versus x at $u$ versus $x$ at $t = 20$. Shown are computational results of the forced Korteweg-de Vries-Burgers' Eq. (1) for $\gamma = 0.05$ (blue solid line) and the forced Korteweg-de Vries equation (red line). The forcing is (8) with $g_0 = -1$ and $W = 0.25$.

hole. Our interest lies in the case of weak dispersion. At this point in the simulation, the solution has not yet reached a steady state. However, if the dispersion is very weak, the solution will reach a quasi-stationary state within the forcing region. This upstream behavior is similar to that observed for positive forcing.

## 3. Comparisons are made between the numerical simulations of viscous and inviscid flow results

In this study, the forced Korteweg-de Vries-Burgers' equation was solved numerically and found to be stable for small values of $\Delta t$, with a truncation error

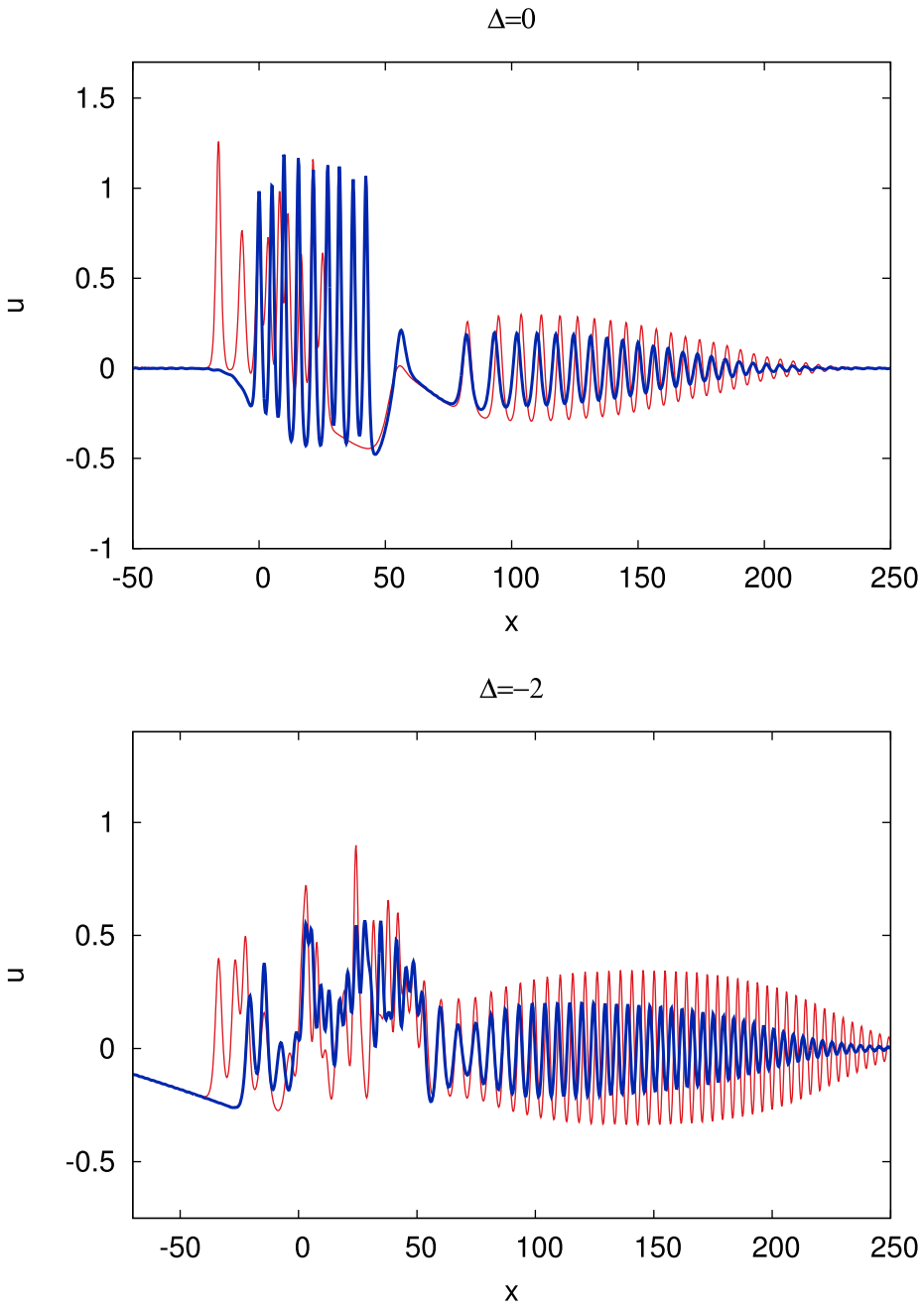


**Fig. 5.** The elevation $u$ versus $x$ at $t = 50$. Shown are computational results of the forced Korteweg-de Vries-Burgers' Eq. (1) for $\gamma = 0.01$ (blue solid line) and the forced Korteweg-de Vries equation (red line), (a) a transcritical flow and (b) a subcritical flow. The forcing is (8) with $g_0 = -1$ and $W = 0.25$.

of $O(\Delta t^2, \Delta x^2)$, which is on the order of $O(\Delta x^3)$. The forcing functions used in our numerical simulations are

$$G(x) = \frac{g_0}{2}\left[\tanh(xW) - \tanh((x - x_L)W)\right], \tag{8}$$

where $W$ measures the width of the step when $g_0 > 0$ and $x_L$ is the distance between the rear and front steps. The parameters used in the present work are $|g_0| = -0.1$, $W = \frac{1}{4}$ and $x_L = 50 >> 1$.

Typical values of the Eq. (1) are presented in Fig. 1 and Fig. 2. A perspective view of the computation in the $x$ versus $x - t$ plane (top) and the surface profile $u$ are illustrated. The bathymetry $G$ (bottom) is also shown at $t = 20$, with $\Delta = 0$ and $W = 0.25$. The numerical calculation of the initial condition $u = 0$ is displayed, hence that the flow is at exact transcritical.

Fig. 1 shows for $g_0 = 1$, the solution consists of three parts, a partial undular bore that propagates upstream, a steady hydraulic flow over the topography, and a partial undular bore downstream of the obstacle, see [21].

Fig. 2 shows for $g_0 = -1$, in this scenario, the results vary from steady to unsteady. At this point, a stationary depression forms in the forcing region, accompanied by small visible oscillations. As a result, a solitary wave is carried downstream to compensate for the depression. The disruptions in the flow are broken down into a limited number of solitary waves through the process of dispersion. Overall, the water flow behavior observed in this study closely aligns with the findings of the forced Korteweg-de Vries equation. However, our findings indicate that viscosity has a significant influence on the properties of water waves, such as the widths of the bores and the amplitudes of the upstream and downstream solitary waves.

The numerical computations of the forced Korteweg-de Vries-Burgers' Eq. (1) illustrated in Fig. 3 and in Fig. 4 show that the flow is not steady over the forcing. The upstream-propagating wavetrains are modulated by the interaction with the downstream-propagating wavetrains. This interaction occurs immediately on the downstream side of the hole, which differs from the analogous case of critical flow over a positive obstacle. As a result, shocks are formed in the area where the forcing occurs. This interaction produces two sets of wavetrains: one moves in the opposite direction and reaches the hole at position $x = 0$, while the other travels downstream and reaches the hole at $x = 50$. Both wavetrains interact with their respective holes. The interaction with the front hole at $x = 0$ indicates that only the largest waves in the pattern can pass through, while the smaller ones are reflected. On the other hand, the interaction with the hole at $x = 50$ appears to be relatively weak, as the wave pattern continues almost unchanged after passing through the hole.

The interaction between upstream and downstream-propagating wavetrains differs significantly between a negative obstacle (depression) and a positive obstacle (elevation). In the case of a positive obstacle, a localized steady state forms over the forcing, while in the present case, bores propagate upstream and downstream of the forcing. This constitutes unsteady states resulting from the collision [16].

Fig. 5(a) illustrates the numerical solutions of both models for a critical flow ($\Delta = 0$) at a large value of $t = 50$. The damping parameter is $\gamma = 0.01$. This figure provides a clear example of the significant role that viscosity plays in determining wave properties.

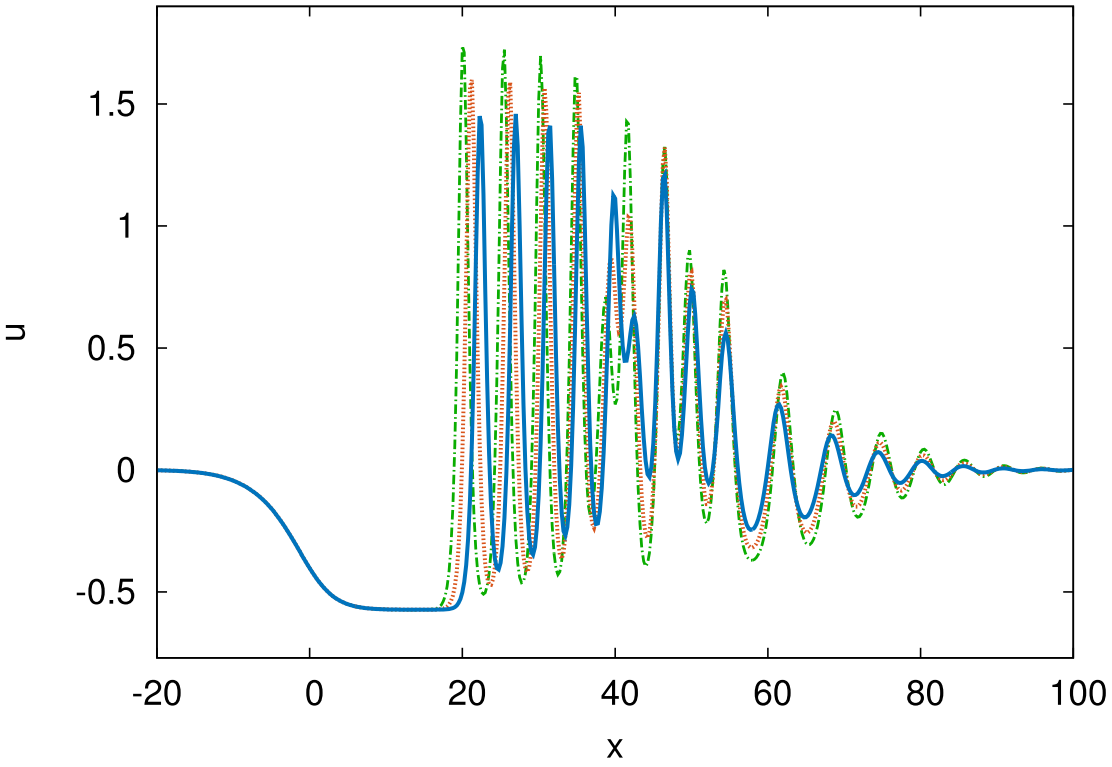


**Fig. 6.** Comparison between the numerical calculations of the forced Korteweg-de Vries equation for $\Delta = 0$ at $t = 20$, (green dashed) and the forced Korteweg-de Vries-Burgers' Eq. (1), for $\gamma = 0.01$ (orange dotted) and for $\gamma = 0.025$ (blue solid line).

Fig. 5(b) compares the two models for a subcritical case, which is similar to the scenarios shown in Fig. 3 but with a longer time frame. The dispersion has a notable impact on the forcing region, affecting the creation of waves both upstream and downstream [16]. Additionally, the solution in the forcing region does not reach a stable or stationary state. In this case, a single shock forms at $x = 0$ and remains stationary. This shock has a positive elevation at $x > 0$ and a negative elevation at $x < 0$. Furthermore, in the region where $x > 0$ and the elevation is positive, dispersion causes the development of a type of wave known as a solitary wave that moves upstream. If this solitary wave does not have enough mass, it will disappear completely once it moves outside the region. In detail, at the case of a positive obstacle, upstream-propagating waves are reflected by the obstacle and converge toward a steady state. For downstream-propagating waves, stable characteristics are formed after interacting with the obstacle. However, upstream waves are improved for a negative obstacle as they are drawn into the low-pressure area. This increases wave steepness and more complex interactions before reaching the obstacle. The downstream waves form variability and instability, as they are affected by the turbulence and acceleration created by the depression. The interaction of waves produces stronger shocks, further accelerating the flow into the depression.

It can be seen from Fig. 6 that the elevation $u$ versus $x$ at $t = 20$ for $\Delta = 0$. The numerical solutions of the forced Korteweg-de Vries equation and the forced Korteweg-de Vries-Burgers' Eq. (1) are compared, for $\gamma = 0.01, 0.025$. As expected, the simulation results of the Korteweg-de Vries model are different than the values of the Korteweg-de Vries-Burgers model.

## 4. Conclusions

In this paper, we have examined the behavior of a fluid flow over a hole in the long wavelength, weakly nonlinear limit. Our model is based on the forced Korteweg-de Vries-Burgers' Eq. (1), which takes into account the damping effect of the fluid's viscosity. We specifically focus on the case of weak damping and present our findings for transcritical, supercritical, and subcritical flows. The results of the numerical simulations for both the viscous and inviscid flow were compared. Our results show that the predictions of the Korteweg-de Vries-Burgers' equation differ from those of the Korteweg-de Vries equation over all the range of flow regimes. Furthermore, we observe that the upstream and downstream amplitudes of the Korteweg-de Vries-Burgers' equation consistently yield lower results compared to the Korteweg-de Vries equation values. For the width of the bores, the Korteweg-de Vries-Burgers' equation predicted narrower bores than the one found in the Korteweg-de Vries equation.

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